Tasho (7/8/2013) Rigid inner forms

motivation

F real ($\mathbb{R}$) / p-adic $\begin{matrix} F \\ | \\ \mathbb{Q}_p \end{matrix}$ $\quad G$: conn. red (q-split) / F $\quad {}^L G = \widehat{G} \rtimes W_F$

$\Gamma = \mathrm{Gal}(\bar{F}/F)$ $\quad W_F$ = abs. Weil gp. $\quad W_F' = L_F = \begin{cases} W_F & F = \mathbb{R} \\ W_F \times SU(2) & F \text{ p-adic} \end{cases}$

$\varphi : L_F \to {}^LG$ L-parameter. **Assume** φ tempered.

$$\begin{array}{ccc} L_F & \xrightarrow{\varphi} & {}^LG \\ & \searrow \quad \# \quad \swarrow & \\ & W_F & \end{array}$$

Expected. $\rightsquigarrow \Pi_\varphi^G$ a finite set of irred adm tempered rep of $G(F)$

Fix a Whittaker datum

$\omega = (B, \psi)$ (Borel, generic $\mathcal{U}(F) \to \mathbb{C}^\times$.)

$S_\varphi = \mathrm{Cent}(\mathrm{Im}\,\varphi, \hat{G})$. Expected: $\Pi_\varphi^G \overset{(1-1)}{\longleftrightarrow} \mathrm{Irr}(\pi_0(S_\varphi / Z(\hat{G})^\Gamma))$

$\pi \longmapsto \langle \pi, - \rangle_\omega$

Conditions

① $\langle \pi, - \rangle_\omega = 1$

$\Leftrightarrow \pi$ is ω-generic

② $\displaystyle\sum_{\pi \in \Pi_\varphi^G} \langle \pi, 1 \rangle_\omega \Theta_\pi$ is stable.

③ If ${}^LH \hookrightarrow {}^LG$ is an endoscopic L-embedding and φ factors thru it

$$\begin{array}{ccc} {}^LH & \hookrightarrow & {}^LG \\ \uparrow & \nearrow & \\ L_F & & \end{array}$$

(H, s)

$$\sum_{\gamma^H} \Delta[\omega](\gamma^H, \gamma) \sum_{\pi \in \Pi_\varphi^H} \langle \pi, 1 \rangle \Theta_\pi(\gamma^H) = \sum_{\pi \in \Pi_\varphi^G} \langle \pi, s \rangle_\omega \Theta_\pi(\gamma)$$

called Langlands-Shelstad transfer factor.

Now we consider "non q-split" G'/F !

$\exists$ a unique G q-split $\xi : G \to G'$ /F, for all $\sigma \in \Gamma$, $\xi^{-1}\sigma(\xi) \in \mathrm{Inn}(G)$ /F.

, ${}^LG' \xrightarrow{\sim} {}^LG$

problem
- No natural normalization of the transfer factor $\Rightarrow$ No parametrization of $\Pi_\varphi^{G'}$
- No Whittaker datum.
- not invariant under aut of endoscopic datum

any normalization is

Other suggestions:
- $\exists$ of add. f.t. ? — spectral transfer factor $\Delta(\varphi, \pi)$.
- mediating f.t. $\rho(\Delta, \xi)$.

over $\mathbb{R}$, Shelstad proved this.

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Vogan : Define Π_φ to consist of iso-classes of tuples (G', ξ, π).
and seek a parameterization.

problem Not possible.

too many auto. $(Sl_2/\mathbb{R}, id, D_k^+) \simeq (Sl_2/\mathbb{R}, id, D_k^-)$.

Solution Enrich each inner twist

$\boxed{(G', \xi)}$ by $z \in Z^1(\Gamma, G)$ with

$\xi^{-1}\sigma(\xi) = Ad(z(\sigma))$.

- possible to define $\Delta[m, z]$ & $\langle -, - \rangle_{m,z}$

problem Not every "(G', ξ)" admits a z.

$$H^1(\Gamma, G) \xrightarrow{\text{not surjective}} H^1(\Gamma, G_{ad}) \longrightarrow H^2(\Gamma, Z(G)).$$

iso classes of pure inner

iso classes of inner.

Let $Z \subset G$. "finite central subgp". $G \to G/Z = \bar{G}$

$\hat{\bar{G}} \to \hat{G}$. "isogeny".

one can define a cohomology class $H^1(u \to w, Z \to G)$ that fits into

$$\begin{array}{ccccc} H^1(\Gamma, G) & \hookrightarrow & H^1(u \to w, Z \to G) & \twoheadrightarrow & H^1(\Gamma, \bar{G}) \\ \downarrow & * & \downarrow & * & \downarrow \\ \pi_0(Z(\hat{G})^\Gamma)^* & \to & \pi_0(Z(\hat{\bar{G}})^+)^* & \to & \pi_0(Z(\hat{\bar{G}})^\Gamma)^* \end{array}$$

bijective if $\begin{cases} F : \text{padic} \\ G : \text{torus} \end{cases}$

What is known: • **$\mathbb{R}$** : 1) diagram + stability + end transfer works for all tempered φ.

2) rigid inner twists $\longleftrightarrow$ strong real forms

• **p-adic** :

$$\sum_{\gamma^H} \underbrace{\Delta[\omega, z](\gamma^H, \gamma')}_{\substack{H?\\ 1 \text{ if } H=G.}} \sum_{\pi \in \Pi_\varphi^H} \langle \pi, 1 \rangle \Theta_\pi(\gamma^H) = \sum_{\pi \in \Pi_\varphi^{G'}} \langle \iota_{\mathfrak{w}}(\xi, z, \pi), \dot{s} \rangle \underbrace{\Theta_\pi(\gamma')}_{\substack{= \\ \text{if } G = H \\ \text{\& split form}}}$$