

03-25-15

Learn Rostani - \mathcal{B}_i

Let G be a connected reductive group over \mathbb{Q}_p
(split, for now).

Bernstein Decomposition:

$$\text{Rep}_{\mathbb{Q}}(G) \cong \prod \mathcal{B}_i$$

\mathcal{B}_i = simple sub categories

index set i = equivalence classes of pairs

(L, σ) ,

L = Levi subgroup of G , σ = fixed super-abelian
rep of L .

The class of (L, σ) is denoted

$[L, \sigma]_G$.

Big Picture Goal: • Find rings R_i such that

$$\mathcal{B}_i \cong R_i\text{-mod}$$

- Understand R_i
- Transport this knowledge back to \mathcal{B}_i . (1)

We ~~will~~ try to answer this question explicitly,
naturally, and in an easy to use way for

Gross Reeder simple supercuspidals. (GRss)

Q: What is a GRss?

Ex: $G = \mathrm{GL}(2)$.

U = pro-unipotent rad. of standard maximal

$$= \begin{pmatrix} 1+\vartheta & 0 \\ p & 1+\vartheta \end{pmatrix}^\times$$

$$U^+ = \begin{pmatrix} 1+\vartheta & \vartheta \\ p^2 & 1+\vartheta \end{pmatrix}^\times$$

$$U/U^+ \cong \mathbb{F}_p \oplus \mathbb{F}_p$$

$$\begin{pmatrix} a & b \\ pc & d \end{pmatrix} \mapsto (b, c)$$

let χ_1, χ_2 be characters of \mathbb{F}_p
(non-trivial)

Define affine generic character χ by the composite
 $U \xrightarrow{\quad} U/U^+ \xrightarrow{\quad} \mathbb{F}_p \oplus \mathbb{F}_p \xrightarrow{\chi_1, \chi_2} \mathbb{C}^\times$

(2)

$\widehat{\pi}_X = \text{cInd}_{Z^G}^G(\pi)$ splits as a direct sum
of supercuspdis, called simple supercuspdis

II. Machinery to help find rings R_i

Bushnell-Kutzko theory of types

~~Defn~~: A type is a pair (\mathcal{T}, ρ) attached
to an index $[L, \sigma]_G$, where \mathcal{T} is
a compact open subgroup, ρ is an irreducible smooth rep of \mathcal{T} ,

Key point:

- 1) If (\mathcal{T}, ρ) is a type for ~~OD~~
 $i = [L, \sigma]_G$, then $\mathcal{H}(G, \mathcal{T}, \rho)$ is ~~OD~~ R_i .
- 2) If σ is known to be ~~odd~~ of
the form $c\text{Ind}_L$, then getting a type
is easy (this is the case $L = G$)
- 3) If $(\tilde{\mathcal{T}}, \tilde{\rho})$ is a "cover" of ~~OD~~
a type (\mathcal{T}, ρ) for $[L, \sigma]_{L^\infty}$, then
 $(\tilde{\mathcal{T}}, \tilde{\rho})$ is a type for $[L, \sigma]_G$.

We construct explicit, natural, covers for

~~the~~ GRss.

Q: What is a ~~or~~ a cover? Need 3 conditions:

- 1) $\tilde{J} \cap L = J$ and $\tilde{f}|_J = f$.
- 2) \tilde{J} should have an Iwahori factorization.
- 3) Technical conditions on existence of invertible elements in $\mathcal{H}(G, \tilde{J}, \tilde{f})$

III.

Prototypical example:

$$G = SL(3), \quad L = GL(2)$$

$$\begin{pmatrix} 11 \\ 2 \times 2 \\ \hline 1 \times 1 \end{pmatrix}$$

σ is ~~or~~ a GRss on L .

Eg: A type for $[L, \sigma]_L$ is (u, x)

$(u, x, x_1, x_2 \text{ from left})$

Lazy Ideq: Try $\tilde{f} = \tilde{U}$, and try to extend

X trivially to a character \tilde{X} on \tilde{U} .

i.e. $\tilde{X}([A]_{**}) = x(A)$.

Is \tilde{X} a character? No

But the failure is not so bad.

$$\tilde{X} \left(\begin{pmatrix} 1+\alpha_p & x & y \\ u_p & 1+\beta_p & z \\ v_p & w_p & 1+\gamma_p \end{pmatrix} \begin{pmatrix} 1+\alpha_p & x & y \\ u_p & 1+\beta_p & z \\ v_p & w_p & 1+\gamma_p' \end{pmatrix} \right)$$

= product of

$$x_1 \left(\cancel{(1+\alpha_p)} (1+\alpha_p)x + x(1+\beta_p) + yw_p \right)$$

and

$$x_2 \left(u(1+\alpha_p) + (1+\beta_p)u + zV \right)$$

Does this equal

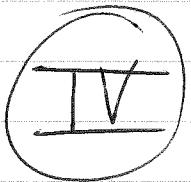
$$\chi_1(x) \chi_2(u) \chi_1(X) \chi_2(v)$$

D No, because of $\chi_2(zV)$.

Solution: increase the depth at the (2,3)
position by 1.

Fact: This works. i.e.

Thm: $(\tilde{\mathfrak{I}}, \tilde{x})$ is a cover.



Key Defn: A root $\alpha \in \mathfrak{I}_G$ is called

L-interfing if α is a positive root and
 $\alpha + H \in \mathfrak{I}_G$ for some

~~highest~~ highest root H of L .

Known's:

1) Explicit description of the fibre in ~~at~~ L of x .

2) $\tilde{J} \cap L = U$, $\tilde{\pi}|_U = \pi$

3) \tilde{J} has Tschirnhaus factorization

4) Invertibility of some elements still in progress.

Note: $\tilde{J}/\tilde{J}_+ \cong U/U_+$, so

Can extend π to \tilde{J} trivially.