ON THE SHARPNESS OF THE BOUND FOR THE LOCAL CONVERSE THEOREM OF p-ADIC GL_N, GENERAL N

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Abstract. Let F be a non-archimedean local field of characteristic zero. In this paper we construct examples of supercuspidal representations showing that the bound $[\frac{N}{2}]$ for the local converse theorem of $\mathrm{GL}_N(F)$ is sharp, N general, when the residual characteristic of F is bigger than N.

1. Introduction

Let F be a non-archimedean local field of characteristic zero. Fix a nontrivial additive character ψ of F. Given irreducible generic representations π and τ of $\mathrm{GL}_N(F)$ and $\mathrm{GL}_r(F)$, respectively, the twisted-gamma factor $\gamma(s, \pi \times \tau, \psi)$ is defined by using Rankin-Selberg convolution ([JPSS83]) or by using Langlands-Shahidi method ([S84]). Fix π, and let τ be any irreducible generic representation of $\mathrm{GL}_r(F)$, $r \geq 1$. The $\gamma(s, \pi \times \tau, \psi)$ give a set of important invariants of π. A natural question to ask is how large should r be in order to completely determine π using these invariants? This is usually called the *Local Converse Problem* for $\mathrm{G}_N = \mathrm{GL}_N$. There is much history to this problem (see [ALST18] for a discussion).

Recently (see [JL16] and [Ch16]), the Jacquet conjecture on the local converse problem for GL_N has been proven, and we have:

Theorem 1.1 ([JL16], [Ch16]). *Let π_1, π_2 be irreducible generic representations of G_N. If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s, for all irreducible generic representations τ of G_r with $r = 1, \ldots, [\frac{N}{2}]$, then $\pi_1 \cong \pi_2$.

By [JNS15, Section 2.4], Theorem 1.1 is shown to be equivalent to the following theorem with the adjective "generic" replaced by "supercuspidal":

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Theorem 1.2. *Let π_1, π_2 be irreducible supercuspidal representations of* G_N. *If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s, for all irreducible supercuspidal representations τ of G_r with $r = 1, \ldots, [\frac{N}{2}]$, then $\pi_1 \cong \pi_2$.

In this paper, we show that the bound $[\frac{N}{2}]$ of r is indeed sharp for Theorem 1.2 when $p > N$, where p is the residual characteristic of F. In previous joint work with Liu, Stevens, and Tam (see [ALST18]), we were only able to show this sharpness result in the case that N is prime.

Precisely, we will construct explicit examples of irreducible supercuspidal representations π_1 and π_2 which are not isomorphic, with the property that

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$$

as functions of the complex variable s, for all irreducible supercuspidal representations τ of G_r with $r = 1, \ldots, [\frac{N}{2}] - 1$.

As in [ALST18], we only need to consider the case that $N \geq 5$. In fact, the examples π_1, π_2 that we construct are precisely the same examples constructed in an earlier unpublished version of [ALST18].

Let us mention the main difference between this paper and [ALST18]. In [ALST18], we assumed that N is prime, and this implied that the relevant Langlands parameters were irreducible. Therefore, we needed to show that a family of gamma factors of irreducible Langlands parameters were equal. In this paper, since N is not necessarily prime, the relevant Langlands parameters are not necessarily irreducible. Therefore, what we need to show is that a family of products of gamma factors of irreducible Langlands parameters are equal. It goes without saying that this paper was heavily influenced by [ALST18].

In Section 2, we recall some background about supercuspidal representations and Langlands parameters for GL_N, and Moy's formula for computing epsilon factors. In Section 3, we construct the examples of supercuspidals which will show that the bound $[\frac{N}{2}]$ is sharp for Theorem 1.2. The main result (Theorem 3.2), that these supercuspidals show that the bound $[\frac{N}{2}]$ is sharp, will be proven in Section 4.

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2. Supercuspidal Representations and Local Langlands Parameters

Let F be a non-archimedean local field of characteristic zero. Let $\mathcal{O}_F$ be the ring of integers of F, $\mathcal{P}_F$ the maximal ideal in $\mathcal{O}_F$, and $\mathbb{F}_q$ the residual field $\mathcal{O}_F/\mathcal{P}_F$ with $q = p^f$ being a power of the residual characteristic p. In this section, we require that $p \nmid N$. Let W_F be the Weil group of F.

In the case of $p \nmid N$, there is a nice parametrization of irreducible representations of W_F of dimension N using admissible quasi-characters introduced by Howe ([Ho77]) as follows.

Definition 2.1 (Howe [Ho77])**.** *Let E/F be an extension of degree N, p doesn't divide N. A quasi-character θ of $E^\times$ is* admissible *with respect to F if*

(i) *θ doesn't come via the norm from a proper subfield of E containing F,*
(ii) *if the restriction $\theta|_{1+\mathcal{P}_E}$ comes via the norm from a subfield $F \subset L \subset E$, then E/L is unramified.*

Two admissible characters θ_1 of E_1/F and θ_2 of E_2/F are said to be conjugate if there is an F-isomorphism between E_1 and E_2 which takes θ_1 to θ_2.

Theorem 2.2 (Moy, Theorem 2.2.2, [M86])**.** *Assume that p doesn't divide N, E/F an extension of degree N. If θ is an admissible character of E/F, then $\mathrm{Ind}_{E/F}\theta$ is an irreducible N-dimensional representation of W_F. Furthermore, two admissible characters induce to equivalent representations if and only if they are conjugate, and each irreducible N-dimensional representation σ of W_F is induced from an admissible character.*

Fix a non-trivial additive character ψ_F of F of level 1, that is, it is trivial on $\mathcal{P}_F$ but not on $\mathcal{O}_F$. For any finite extension E/F, define an additive character ψ_E of E via $\psi_E = \psi_F \circ \mathrm{tr}_{E/F}$. It is known that if E/F is tamely ramified, then ψ_E is also of level 1, that is, it is trivial on $\mathcal{P}_E$ but not on $\mathcal{O}_E$. Let U_F be the group of roots of unity in $F^\times$ of order prime to p. Fix a uniformizer ϖ_F of F and let C_F be the group generated by U_F and ϖ_F. For any finite extension E/F, C_E can be defined similarly, which is uniquely determined by ϖ_F. The norm map $N_{E/F}$ takes C_E into C_F.

Let E/F be a tamely ramified extension of degree N, and θ a quasi-character of $E^\times$. Let $c = f_E(\theta)$ be the conductoral exponent of θ, that is, θ is trivial on $1 + \mathcal{P}_E^c$ but not on $1 + \mathcal{P}_E^{c-1}$. If $c > 1$, then there is a unique element $\gamma_\theta \in C_E$ such that $\theta(1+x) = \psi_E(\gamma_\theta x)$ for $x \in \mathcal{P}_E^{c-1}$. γ_θ is called the standard representative of θ, and we say that γ_θ *represents* θ. Note that $\mathrm{val_E}(\gamma_\theta) = 1 - \mathrm{c}$, and if K/E is a finite extension and $\phi = \theta \circ N_{K/E}$, then $\gamma_\phi = \gamma_\theta$.

Definition 2.3 (Kutzko, [Ku80])**.** *Let E/F be an extension of degree N, p doesn't divide N. A quasi-character θ of $E^\times$ is* generic *over F if either*

(i) *$f_E(\theta) = 1$, E/F is unramified and θ doesn't come via the norm from a proper subfield of E containing F, or*
(ii) *$f_E(\theta) > 1$, and $F(\gamma_\theta) = E$.*

Note that a generic character is necessarily admissible. If $F \subset L \subset E$ and θ is a quasi-character of $L^\times$, let $\theta_E = \theta \circ N_{E/L}$, or simply θ when the context is clear. Viewed as a character of W_L, θ_E is the restriction of θ to the subgroup W_E.

Lemma 2.4 (Howe's Factorization Lemma, [Ho77] and [M86])**.** *Let E/F be an extension of degree N, p doesn't divide N. Let θ be an admissible character of E/F. Then* *there is a unique tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_s = E$ and quasi-characters $\chi = \phi_0, \phi_1, \ldots, \phi_s$ of $F^\times = F_0^\times, F_1^\times, \ldots, F_s^\times = E^\times$ respectively, with ϕ_k generic over F_{k-1}, $k = 1, 2, \ldots, s$, such that*

$$\theta = \chi\phi_1\phi_2\cdots\phi_s.$$

ϕ_k's are well-defined modulo a character coming from $F_{k-1}^\times$ via the norm map, and the conductoral exponents $f_E(\phi_1) > f_E(\phi_2) > \cdots > f_E(\phi_s)$ are unique.

In [Ho77], Howe constructed supercuspidal representations of $\mathrm{G}_N(F)$ from admissible characters of $E^\times$ with respect to F when E/F is tamely ramified of degree N. In [M86], Moy showed that Howe's construction exhausted all the supercuspidal representations in this case. Given an admissible character of $E^\times$, denote by π_θ the supercuspidal representation constructed by Howe.

In [M86], Moy also computed the ϵ-factors of θ as follows. Let θ a quasi-character of $F^\times$. Let $c = f(\theta)$ be the conductoral exponent of θ, that is, θ is trivial on $1 + \mathfrak{p}_F^c$ but not on $1 + \mathfrak{p}_F^{c-1}$.

When $f(\theta) > 1$, let $r = [\frac{f(\theta)+1}{2}]$. Define $c_\theta \in \mathfrak{p}_F^{1-f(\theta)}$ mod $\mathfrak{p}_{\mathrm{F}}^{1-\mathrm{r}}$ such that

$$\theta(1 + x) = \psi(c_\theta x),$$

for $x \in \mathfrak{p}_F^r$. If E/F is a finite extension, then $c_{\theta\circ N_{E/F}} = c_\theta$.

Suppose $f(\theta) = f_F(\theta) = 2n + 1$ is odd. Set $H = U_F^n$, $J = U_F^{n+1}$. We define the Gauss sum

$$G(\theta, \psi) = q^{-\frac{1}{2}} \sum_{x \in H/J} \theta^{-1}(x)\psi_F(c_\theta(x - 1)).$$

Denote by $\epsilon(\theta, \psi)$ the value of the ϵ-factor at $s = 0$. Note that $\epsilon(s, \theta, \psi) = \epsilon(0, \theta|\cdot|^s, \psi)$.

Theorem 2.5 (Moy, [M86], (2.3.17))**.**

(i) *Suppose $f(\theta)$ is even. Then*

$$\epsilon(\theta, \psi) = \theta^{-1}(c_\theta)\psi(c_\theta)|c_\theta|^{1/2}.$$

(ii) *Suppose $f(\theta)$ is odd. Then*

$$\epsilon(\theta, \psi) = \theta^{-1}(c_\theta)\psi(c_\theta)|c_\theta|^{1/2}G(\theta, \psi).$$

3. Construction

In this section, we explicitly construct two non-isomorphic supercuspidal representations of $G_N(F)$ which will later show that the bound $[\frac{N}{2}]$ of r in Conjecture 1.2 is sharp. From now until the end of the paper, we assume that $p > N$.

Fix a uniformizer ϖ_F in F, and define a totally ramified extension $E = F[\sqrt[N]{\varpi_F}]$. Since we assume that $p > N$, in particular, $p \nmid N$, E is a tamely ramified extension of F of degree N. Set $\varpi_E := \sqrt[N]{\varpi_F}$.

Recall that we have assumed that $N \geq 5$. The element $\beta := \varpi_E^{2-2N}$ defines a character of $1+\mathcal{P}_E^{2N-2}$, trivial on $1+\mathcal{P}_E^{2N-1}$. When N is odd, we extend this character in any way to $E^\times$, and denote the resulting extension by ϕ. That ϕ is admissible follows readily since one can see that it is generic (see Definition 2.3). When N is even, the element $\beta_1 = \beta = \varpi_E^{2-2N}$ is in C_{E_1}, where $E_1 = F[\beta_1] \subsetneq E$. We use the same construction as in the odd case to obtain a character ϕ_1 on $E_1^\times$. We then choose an integer $\ell \in [2, N-1]$ that is coprime to N and define $\beta_2 = \varpi_E^{-\ell}$. This condition implies that $E = E_1[\beta_2]$, and $\beta_2 \in C_E$. Similar to above, we define a character ϕ_2 on $1+\mathcal{P}_E^{\ell}$ trivial on $1+\mathcal{P}_E^{\ell+1}$ by β_2, and extend it to a character, still denoted by ϕ_2, on $E^\times$. Now take

$$\phi = (\phi_1 \circ N_{E/E_1})\phi_2.$$

The above constructions imply that ϕ is admissible over F, and that the product above is the Howe factorization of ϕ. We emphasize that we have now defined a character ϕ, which has a different definition when N is odd from when N is even.

We now define two characters $\phi^{(1)}, \phi^{(2)}$, on $E^\times$ by first setting $\phi^{(1)}(\varpi_E) = \phi^{(2)}(\varpi_E)$ and $\phi^{(1)}|_{k_F^\times} \equiv \phi^{(2)}|_{k_F^\times}$. We then set $\phi^{(i)}|_{1+\mathcal{P}_E^2} \equiv \phi|_{1+\mathcal{P}_E^2}$, and then extend $\phi^{(i)}|_{1+\mathcal{P}_E^2}$ to $1+\mathcal{P}_E$ in two different ways to produce two characters $\phi^{(1)}, \phi^{(2)}$, that differ on $(1+\mathcal{P}_E) \setminus (1+\mathcal{P}_E^2)$.

We observe that $\phi^{(1)}$ and $\phi^{(2)}$ can be chosen in a way as to not be isomorphic as admissible pairs. Indeed, there are $q-1$ nontrivial characters of $1+\mathcal{P}_E$ that are trivial on $1+\mathcal{P}_E^2$. However, there are at most $[E:F]$ admissible characters of $E^\times$ that are conjugate to any given admissible character of $E^\times$ (since two characters of $E^\times$ are isomorphic if and only if there exists an F-automorphism of E taking one character to the other). But $[E:F] = N$, and we have assumed from the beginning that $p > N$.

Remark 3.1. *The requirements that we have placed on $\phi^{(1)}, \phi^{(2)}$ imply $\phi^{(1)}|_{F^\times} \equiv \phi^{(2)}|_{F^\times}$ since $\phi^{(1)}(\varpi_E) = \phi^{(2)}(\varpi_E), \phi^{(1)}|_{k_F^\times} \equiv \phi^{(2)}_{k_F^\times}$, and $1+\mathcal{P}_F \subset 1+\mathcal{P}_E^2$. We will need this later.*

Set $\pi_i = \pi_{\phi^{(i)}}$. We see by the above construction that $\pi_1 \ncong \pi_2$. The following is our main result, showing that the bound $[\frac{N}{2}]$ of r in Conjecture 1.2 is indeed sharp.

Theorem 3.2.

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi), \tag{3.1}$$

as functions of the complex variable s, for all irreducible supercuspidal representations τ of G_r with $r = 1, \ldots, [\frac{N}{2}] - 1$.

4. Proof of Theorem 3.2

Henceforth, if T/F is an extension, we denote by $\mathrm{Gal}(T/F)$ the set of all embeddings of T into $\overline{F}$ that fix F, whether or not T/F is Galois. Moreover, for a character χ of $F^\times$, we denote by $\chi_{T/F}$ the character $\chi \circ N_{T/F}$.

Proof of Theorem 3.2. Let τ be any irreducible supercuspidal representation of G_r, $1 \leq r \leq [\frac{N}{2}] - 1$. Since $p \nmid r$, by the discussion in Section 2, one may assume that the local Langlands parameter of τ is $\mathrm{Ind}_{W_L}^{W_F} \lambda$, for some admissible pair $(L/F, \lambda)$. Let $f_L(\lambda)$ be the conductoral exponent of λ, and let $m = f_L(\lambda) - 1$, so that there exists an element $\alpha \in \mathcal{P}_L^{-m}$ mod $\mathcal{P}_L^{-[\frac{m}{2}]}$ that represents λ, that is,

$$\lambda(1 + x) = \psi_L(\alpha x),$$

for $x \in \mathcal{P}_L^{[\frac{m}{2}]+1}$. Let $e_L = e(L/F)$ be the ramification index of L/F. Finally, let M be the Galois closure of $K = EL$, over F.

To prove (3.1), it suffices by the local Langlands correspondence for GL_N to prove that

$$\gamma(s, \mathrm{Ind}_{W_E}^{W_F} \phi^{(1)} \otimes \mathrm{Ind}_{W_L}^{W_F} \lambda, \psi) = \gamma(s, \mathrm{Ind}_{W_E}^{W_F} \phi^{(2)} \otimes \mathrm{Ind}_{W_L}^{W_F} \lambda, \psi) \tag{4.1}$$

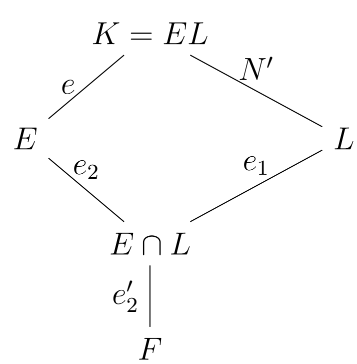
for all such $(L/F, \lambda)$.

Let $\theta_g^{(i)} = [({}^g\phi^{(i)} \circ N_{L({}^gE)/{}^gE}) \otimes (\lambda \circ N_{L({}^gE)/L})]$, for $g \in W_L \backslash W_F / W_E$, and write $K_g = L({}^gE)$, $E_g = {}^gE$. We note that ${}^g\beta + \alpha$ represents $\theta_g^{(i)}$, for both $i = 1, 2$.

By Lemma 4.1, 4.2, 4.3, one has to show that

$$\prod_{g \in W_L \backslash W_F / W_E} {}^g\phi^{(1)} \circ N_{K_g/E_g}({}^g\beta + \alpha) = \prod_{g \in W_L \backslash W_F / W_E} {}^g\phi^{(2)} \circ N_{K_g/E_g}({}^g\beta + \alpha) \tag{4.2}$$

We separate the proof of (4.2) into three cases: (1) $val_K(\beta) = val_K(\alpha)$; (2) $val_K(\beta) < val_K(\alpha)$; (3) $val_K(\beta) > val_K(\alpha)$, and we note that $val_{K_g}({}^g\beta) = val_K(\beta), val_{K_g}(\alpha) = val_K(\alpha)$ $\forall g \in W_L \backslash W_F / W_E$. Let $e = e(K/E)$ and $N' = e(K/L)$. We have a diagram of fields, with ramification indices listed on the line segments.



We note in particular that since E/F is totally ramified, we have that $N = e_2e_2'$. We also have $\beta \in \mathcal{P}_K^{-e(2N-2)}$ and $\alpha \in \mathcal{P}_K^{-mN'}$.

Case (1). $val_K(\beta) = val_K(\alpha)$. Then, $e(2N-2) = mN'$. Since E/F is totally ramified, and by the diamond, we have that $N' = \frac{ee_2}{e_1}$. Altogether, the equality $e(2N-2) = mN'$ can be rewritten now as

$$2e_2e_2' - 2 = m\frac{e_2}{e_1}.$$

Multiplying both sides of the equality by e_1e_2' yields

$$2Ne_1e_2' - 2e_1e_2' = mN.$$

Simplifying, we get $N(2e_1e_2' - m) = 2e_1e_2' = 2e_L$. Therefore, we obtain that N divides $2e_L$. On the other hand, we have $1 \leq e_L \leq r < \frac{N-1}{2}$, a contradiction.

Case (2). $val_K(\beta) < val_K(\alpha)$. Since ${}^g\phi^{(1)}_{K_g/E_g}({}^g\beta) = {}^g\phi^{(2)}_{K_g/E_g}({}^g\beta)$, we just have to show that

$$\prod_{g \in W_L\backslash W_F/W_E} {}^g\phi^{(1)} \circ N_{K_g/E_g}(1 + ({}^g\beta)^{-1}\alpha) = \prod_{g \in W_L\backslash W_F/W_E} {}^g\phi^{(2)} \circ N_{K_g/E_g}(1 + ({}^g\beta)^{-1}\alpha) \tag{4.3}$$

We compute the left hand side.

$$\begin{aligned}
&\prod_{g \in W_L\backslash W_F/W_E} {}^g\phi^{(1)} \circ N_{K_g/E_g}(1 + ({}^g\beta)^{-1}\alpha) \\
&= \phi^{(1)}\left(\prod_{g \in W_L\backslash W_F/W_E}\left(g^{-1}\left(\prod_{\sigma \in \mathrm{Gal}(K_g/E_g)}(1 + \sigma(({}^g\beta^{-1})\alpha))\right)\right)\right) \\
&= \phi^{(1)}\left(\prod_{g \in W_L\backslash W_F/W_E}\prod_{\sigma \in \mathrm{Gal}(K_g/E_g)} g^{-1}(1 + {}^g\beta^{-1}\sigma(\alpha))\right) \\
&= \phi^{(1)}\left(\prod_{g \in W_L\backslash W_F/W_E}\left(\prod_{\sigma \in \mathrm{Gal}(K_g/E_g)}(1 + \beta^{-1}(g^{-1}\sigma)(\alpha))\right)\right)
\end{aligned}$$

since ${}^g\beta^{-1} \in E_g$ is fixed by all σ.

We now note that

$$W_L\backslash W_F/W_E \cong W_E\backslash W_F/W_L \cong Gal(M/E)\backslash Gal(M/F)/Gal(M/L),$$

where M is the Galois closure of the compositum EL (see [ALST18, Lemma 2.6]. We obtain that $W_L\backslash W_F/W_E$ can be interpreted as the orbits in $Gal(M/F)/Gal(M/L) = Gal(L/F)$, under the action of $Gal(M/E)$. To view this action, we can think of $Gal(L/F)$ as embeddings of L into M (since M is the Galois closure of EL) that fix F.

We are therefore considering a set of elements $g^{-1}\sigma$, where g ranges over a set of representatives of the orbits of $Gal(M/E)$ acting on $Gal(L/F)$, and $\sigma \in Gal(K_g/E_g)$. We wish to show that this set of $g^{-1}\sigma$ exhausts all of $Gal(L/F)$. In fact it is better to write $g^{-1}\sigma$ as $(g^{-1}\sigma g)g^{-1}$. We will show that these elements exhaust all of $Gal(L/F)$.

So we now prove that

$$\{g^{-1}\sigma : g \in Gal(M/E)\backslash Gal(M/F)/Gal(M/L), \sigma \in \mathrm{Gal}(K_g/E_g)\}$$

can be identified canonically with $Gal(L/F)$, which would simplify the above expression to

$$\phi^{(1)}\left(\prod_{\sigma \in \mathrm{Gal}(L/F)} (1+\beta^{-1}\sigma(\alpha))\right).$$

Let $H = Gal(M/E), G = Gal(M/F), J = Gal(M/L)$, so that H acts on G/J. We consider an orbit $Hg^{-1}J, g \in G$. By the orbit-stabilizer theorem, $Hg^{-1}J \cong \frac{H}{\mathrm{Stab}_H(g^{-1}J)}$. Furthermore, we argue that $\mathrm{Stab}_H(g^{-1}J) \cong Gal(M/(g^{-1}K_g))$. Indeed, first note that $x \in Gal(M/E)$ stabilizes $g^{-1}J$ if and only if $xg^{-1}J = g^{-1}J$, i.e. that $x \in g^{-1}Jg = Gal(g^{-1}M/g^{-1}L) = Gal(M/g^{-1}L)$, since $g^{-1}M = M$. Therefore,

$$\begin{aligned} & x \in \mathrm{Stab}_H(g^{-1}J) \\ \iff & x \in Gal(M/E) \cap Gal(M/(g^{-1}L)) \\ \iff & x \in Gal(M/((g^{-1}L)E) = Gal(M/(g^{-1}K_g)) \end{aligned}$$

We conclude that $Hg^{-1}J \cong \frac{H}{\mathrm{Stab}_H(g^{-1}J)} \cong \frac{Gal(M/E)}{Gal(M/(g^{-1}K_g))} \cong Gal((g^{-1}K_g)/E)$, noting that $g^{-1}K_g = (g^{-1}L)E$. Now, via $\eta \mapsto g\eta g^{-1}$, we have $Gal((g^{-1}K_g)/E) \cong Gal(K_g/E_g)$. The identifications above now allow us to conclude an isomorphism $Gal(K_g/E_g) \xrightarrow{\sim} \frac{H}{\mathrm{Stab}_H(g^{-1}J)}$ given by σ maps to the class of $g^{-1}\sigma g$, with $\sigma \in Gal(K_g/E_g)$. Therefore, the orbit $Hg^{-1}J$ can be identified with the set of $(g^{-1}\sigma g)g^{-1}J$ such that $\sigma \in Gal(K_g/E_g)$, i.e. the set of $g^{-1}\sigma J$, which is exactly what we set out to prove.

So we obtain

$$\phi^{(1)}\left(\prod_{\sigma \in \mathrm{Gal}(L/F)} (1+\beta^{-1}\sigma(\alpha))\right).$$

We now have

$$\phi^{(1)}\left(\prod_{\sigma \in \mathrm{Gal}(L/F)} (1+\beta^{-1}\sigma(\alpha))\right) = \phi^{(1)}\left(1 + \sum_{i=1}^{s} \beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha)\right)$$

where $s = \#Gal(L/F)$, and where B_i is the set of all i-element subsets of $Gal(L/F)$.

Claim: $P(\alpha) := \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \in F$. To see this, firstly, the sum is over $Gal(L/F)$. By construction, $P(\alpha)$ is fixed by every embedding h in $Gal(\overline{L}/F)$ because composition with h just permutes the embeddings in $Gal(L/F)$, so h just permutes the terms in the sum giving $P(\alpha)$.

Since $\beta^{-1} \in \mathcal{P}_E^{2N-2}$, and $\alpha \in \mathcal{P}_L^{-m}$, we have

$$\beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \in \mathcal{P}_E^{i(2N-2)+e_2 e_2' \lceil \frac{-im}{e_1 e_2'} \rceil},$$

noting that $\mathcal{P}_L^{-im} \cap F = \mathcal{P}_F^{\lceil \frac{-im}{e_1 e_2'} \rceil}$ (this is where we use the above claim that $P(\alpha) \in F$).

We write

$$A := i(2N-2) + e_2 e_2' \lceil \frac{-im}{e_1 e_2'} \rceil = i(2N-2) + N \lceil \frac{-im}{e_1 e_2'} \rceil$$

Modulo N, we get

$$A \equiv -2i \ (mod\ N),$$

Since $1 \leq i \leq s$, we have that, modulo N, $2 \leq -A \leq 2s$. Note also that $1 \leq r < \frac{N-1}{2}$. Therefore, we have that, modulo N, $2 \leq -A \leq 2s \leq 2r < N-1$. In particular, one now sees that $A \not\equiv 1$ mod N. In particular, $A \neq 1$. Hence, $A \geq 2$, and $1 + \sum_{i=1}^{s} \beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \in 1 + \mathcal{P}_E^2$. Therefore, since $\phi^{(1)}, \phi^{(2)}$ agree on $1 + \mathcal{P}_E^2$, we have

$$\phi^{(1)} \left(1 + \sum_{i=1}^{s} \beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \right) = \phi^{(2)} \left(1 + \sum_{i=1}^{s} \beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \right),$$

completing the proof of Case (2).

Case (3). $val_K(\beta) > val_K(\alpha)$. This proof here is similar to the proof of case (2). We note that

$$\begin{aligned}
&\prod_{g \in W_L \backslash W_F / W_E} {}^g\phi^{(i)} \circ N_{K_g/E_g}(\alpha) \\
&= \phi^{(i)} \left(\prod_{g \in \mathrm{Gal}(M/E) \backslash \mathrm{Gal}(L/F)} \prod_{\sigma \in \mathrm{Gal}(K/E)} \sigma(\alpha)) \right) \\
&= \phi^{(i)} \left(\prod_{g \in \mathrm{Gal}(K/E) \backslash \mathrm{Gal}(L/F)} \prod_{\sigma \in \mathrm{Gal}(K/E)} \sigma(\alpha)) \right) \\
&= \phi^{(i)} \left(\prod_{\sigma \in \mathrm{Gal}(L/F)} \sigma(\alpha) \right) \\
&= \phi^{(i)}(N_{L/F}(\alpha)),
\end{aligned}$$

and since $\phi^{(1)}|_{F^\times} \equiv \phi^{(2)}|_{F^\times}$ (see Remark 3.1), we see that (4.2) is equivalent to

$$\prod_{g \in W_L \backslash W_F / W_E} {}^g\phi^{(1)} \circ N_{K_g/E_g}(1 + {}^g\beta\alpha^{-1}) = \prod_{g \in W_L \backslash W_F / W_E} {}^g\phi^{(2)} \circ N_{K_g/E_g}(1 + {}^g\beta\alpha^{-1})$$

But the argument now is the same is in case (2). On the left, we arrive at the term

$$\phi^{(1)} \left(\prod_{\sigma \in \mathrm{Gal}(L/F)} (1 + \beta\sigma(\alpha^{-1})) \right),$$

and also the term (using the same notation as in case (2)) $A = -(i(2N-2) + e_2e_2'\lceil \frac{-im}{e_1e_2'} \rceil)$, which is precisely the negative of the term A that appeared in case (2). We still have $A > 0$, since $val_K(\beta) > val_K(\alpha)$. Then, $A \equiv 2i \ (mod\ N)$, and $2 \leq A < N-1$, so that $A \not\equiv 1 \bmod N$ and thus $A \neq 1$. Hence, we have proven Case (3).

This completes the proof of Theorem 3.2, up to the the proofs of Lemmas 4.1, 4.2, and 4.3. □

4.1. **Lemmas.** In this section, we lay out the various Lemmas that we needed in the proof of Theorem 3.2.

Lemma 4.1. [ALST18, Lemma 2.5] *For* $i = 1, 2$, $(Ind_{W_E}^{W_F} \phi^{(i)}) \otimes (Ind_{W_L}^{W_F} \lambda)$ *is isomorphic to*

$$\bigoplus_{g \in W_L \backslash W_F / W_E} Ind_{W_{L({}^gE)}}^{W_F} [({}^g\phi^{(i)} \circ N_{L({}^gE)/{}^gE}) \otimes (\lambda \circ N_{L({}^gE)/L})].$$

Recall that $\theta_g^{(i)} = [({}^g\phi^{(i)} \circ N_{L({}^gE)/{}^gE}) \otimes (\lambda \circ N_{L({}^gE)/L})]$, for $g \in W_L \backslash W_F / W_E$, and $K_g = L({}^gE)$, $E_g = {}^gE$. We also write $\nu_g^{(i)} = [({}^g\phi^{(i)} \circ N_{M_g/E_g}) \otimes (\lambda \circ N_{M_g/L})]$ for $i = 1, 2$.

Since the L-function is inductive, multiplicative, and trivial on ramified characters, and by [ALST18, Lemma 2.4], we have

Lemma 4.2. *We have for $i = 1, 2$,*

$$\gamma(s, \bigoplus_{g \in W_L \backslash W_F / W_E} \mathrm{Ind}_{W_{K_g}}^{W_F} \theta_g^{(i)}, \psi_F) = \prod_{g \in W_L \backslash W_F / W_E} \lambda_{K_g/F}(\psi_F) \gamma(s, \theta_g^{(i)}, \psi_{K_g}),$$

where $\lambda_{K_g/F}(\psi_F)$ is the Langlands constant associated to K_g/F and ψ_F (see [BH06, Theorem 29.4]*,* [BH06, §34.3]*), and $\psi_{K_g} = \psi_F \circ \mathrm{tr}_{K_g/F}$. The formula is also true when γ is replaced by ϵ.*

Lemma 4.3.

$$\frac{\gamma(s, \theta_g^{(1)}, \psi_{K_g})}{\gamma(s, \theta_g^{(2)}, \psi_{K_g})} = \frac{\theta_g^{(1)}({}^g\beta + \alpha)}{\theta_g^{(2)}({}^g\beta + \alpha)}$$

Proof. It suffices to prove the formula with γ replaced by ϵ, by the proof of Lemma 4.2.

We first note that $c_{\nu_g^{(1)}} = c_{\nu_g^{(2)}} = {}^g\beta + \alpha$ so that in particular $f := f_{K_g}(\nu_g^{(1)}) = f_{K_g}(\nu_g^{(2)})$. If f is even, then the the proposed equality is easily seen to be true by Theorem 2.5.

Suppose that $f = 2n + 1$ is odd. We must check that $G(\theta_g^{(1)}, \psi_{K_g}) = G(\theta_g^{(2)}, \psi_{K_g})$. The impact of $g \in W_L \backslash W_F / W_E$ is negligible, so we will assume that $g = 1$.

Set $\theta^{(i)} = \phi^{(i)} \circ N_{LE/E} \otimes \lambda \circ N_{LE/L}$, $K = EL$, and $f = f_K(\theta^{(1)}) = f_K(\theta^{(2)}) = 2n+1$ is odd. Since $c_{\theta^{(1)}} = c_{\theta^{(2)}} = \beta + \alpha$, and by definition of $\theta^{(i)}$, it suffices to check that

$$(\phi^{(1)} \circ N_{K/E})(x) = (\phi^{(2)} \circ N_{K/E})(x) \text{ for } x \in U_K^n = 1 + \mathcal{P}_K^n. \tag{4.4}$$

We first note that $f = \max\{e(2N-2), mN'\} + 1$. Therefore, since $N \geq 5$, we have that $n \geq \frac{3}{2} > e$, so that $N_{K/E}(U_K^n) \subset U_E^2$ by a standard property of the norm map. Since $\phi^{(1)}$ and $\phi^{(2)}$ agree on U_E^2, we have that (4.4) is proven. ☐

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