ON THE SHARPNESS OF THE BOUND FOR THE LOCAL CONVERSE THEOREM OF $p$-ADIC $GL_N$, GENERAL $N$

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Abstract. Let $F$ be a non-archimedean local field of characteristic zero. In this paper we construct examples of supercuspidal representations showing that the bound $\left\lceil \frac{N}{2} \right\rceil$ for the local converse theorem of $GL_N(F)$ is sharp, $N$ general, when the residual characteristic of $F$ is bigger than $N$.

1. Introduction

Let $F$ be a non-archimedean local field of characteristic zero. Fix a nontrivial additive character $\psi$ of $F$. Given irreducible generic representations $\pi$ and $\tau$ of $GL_N(F)$ and $GL_r(F)$, respectively, the twisted-gamma factor $\gamma(s, \pi \times \tau, \psi)$ is defined by using Rankin-Selberg convolution ([JPSS83]) or by using Langlands-Shahidi method ([SS84]). Fix $\pi$, and let $\tau$ be any irreducible generic representation of $GL_r(F)$, $r \geq 1$. The $\gamma(s, \pi \times \tau, \psi)$ give a set of important invariants of $\pi$. A natural question to ask is how large should $r$ be in order to completely determine $\pi$ using these invariants? This is usually called the Local Converse Problem for $G_N = GL_N$. There is much history to this problem (see [ALST18] for a discussion).

Recently (see [JL16] and [Ch16]), the Jacquet conjecture on the local converse problem for $GL_N$ has been proven, and we have:

**Theorem 1.1** ([JL16], [Ch16]). Let $\pi_1, \pi_2$ be irreducible generic representations of $G_N$. If

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable $s$, for all irreducible generic representations $\tau$ of $G_r$ with $r = 1, \ldots, \left\lceil \frac{N}{2} \right\rceil$, then $\pi_1 \cong \pi_2$.

By [JNSI15, Section 2.4], Theorem 1.1 is shown to be equivalent to the following theorem with the adjective “generic” replaced by “supercuspidal”:

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Theorem 1.2. Let $\pi_1, \pi_2$ be irreducible supercuspidal representations of $G_N$. If
\[
\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),
\]
as functions of the complex variable $s$, for all irreducible supercuspidal representations $\tau$ of $G_r$ with $r = 1, \ldots, \lfloor \frac{N}{2} \rfloor$, then $\pi_1 \cong \pi_2$.

In this paper, we show that the bound $\lfloor \frac{N}{2} \rfloor$ of $r$ is indeed sharp for Theorem 1.2 when $p > N$, where $p$ is the residual characteristic of $F$. In previous joint work with Liu, Stevens, and Tam (see [ALST18]), we were only able to show this sharpness result in the case that $N$ is prime.

Precisely, we will construct explicit examples of irreducible supercuspidal representations $\pi_1$ and $\pi_2$ which are not isomorphic, with the property that
\[
\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)
\]
as functions of the complex variable $s$, for all irreducible supercuspidal representations $\tau$ of $G_r$ with $r = 1, \ldots, \lfloor \frac{N}{2} \rfloor - 1$.

As in [ALST18], we only need to consider the case that $N \geq 5$. In fact, the examples $\pi_1, \pi_2$ that we construct are precisely the same examples constructed in an earlier unpublished version of [ALST18].

Let us mention the main difference between this paper and [ALST18]. In [ALST18], we assumed that $N$ is prime, and this implied that the relevant Langlands parameters were irreducible. Therefore, we needed to show that a family of gamma factors of irreducible Langlands parameters were equal. In this paper, since $N$ is not necessarily prime, the relevant Langlands parameters are not necessarily irreducible. Therefore, what we need to show is that a family of products of gamma factors of irreducible Langlands parameters are equal. It goes without saying that this paper was heavily influenced by [ALST18].

In Section 2, we recall some background about supercuspidal representations and Langlands parameters for $GL_N$, and Moy’s formula for computing epsilon factors. In Section 3, we construct the examples of supercuspidals which will show that the bound $\lfloor \frac{N}{2} \rfloor$ is sharp for Theorem 1.2. The main result (Theorem 3.2), that these supercuspidals show that the bound $\lfloor \frac{N}{2} \rfloor$ is sharp, will be proven in Section 4.

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2. Supercuspidal Representations and Local Langlands Parameters

Let $F$ be a non-archimedean local field of characteristic zero. Let $\mathcal{O}_F$ be the ring of integers of $F$, $\mathcal{P}_F$ the maximal ideal in $\mathcal{O}_F$, and $\mathbb{F}_q$ the residual field $\mathcal{O}_F/\mathcal{P}_F$ with $q = p^f$ being a power of the residual characteristic $p$. In this section, we require that $p \nmid N$. Let $W_F$ be the Weil group of $F$. 
In the case of $p \nmid N$, there is a nice parametrization of irreducible representations of $W_F$ of dimension $N$ using admissible quasi-characters introduced by Howe (\cite{Ho77}) as follows.

**Definition 2.1** (Howe \cite{Ho77}). Let $E/F$ be an extension of degree $N$, $p$ doesn’t divide $N$. A quasi-character $\theta$ of $E^\times$ is admissible with respect to $F$ if

(i) $\theta$ doesn’t come via the norm from a proper subfield of $E$ containing $F$,

(ii) if the restriction $\theta|_{1+P_E}$ comes via the norm from a subfield $F \subset L \subset E$, then $E/L$ is unramified.

Two admissible characters $\theta_1$ of $E_1/F$ and $\theta_2$ of $E_2/F$ are said to be conjugate if there is an $F$-isomorphism between $E_1$ and $E_2$ which takes $\theta_1$ to $\theta_2$.

**Theorem 2.2** (Moy, Theorem 2.2.2, \cite{M86}). Assume that $p$ doesn’t divide $N$, $E/F$ an extension of degree $N$. If $\theta$ is an admissible character of $E/F$, then $\text{Ind}_{E/F}\theta$ is an irreducible $N$-dimensional representation of $W_F$. Furthermore, two admissible characters induce to equivalent representations if and only if they are conjugate, and each irreducible $N$-dimensional representation $\sigma$ of $W_F$ is induced from an admissible character.

Fix a non-trivial additive character $\psi_F$ of $F$ of level 1, that is, it is trivial on $P_F$ but not on $O_F$. For any finite extension $E/F$, define an additive character $\psi_E$ of $E$ via $\psi_E = \psi_F \circ \text{tr}_{E/F}$. It is known that if $E/F$ is tamely ramified, then $\psi_E$ is also of level 1, that is, it is trivial on $P_E$ but not on $O_E$. Let $U_F$ be the group of roots of unity in $F^\times$ of order prime to $p$. Fix a uniformizer $\varpi_F$ of $F$ and let $C_F$ be the group generated by $U_F$ and $\varpi_F$. For any finite extension $E/F$, $C_E$ can be defined similarly, which is uniquely determined by $\varpi_F$. The norm map $N_{E/F}$ takes $C_E$ into $C_F$.

Let $E/F$ be a tamely ramified extension of degree $N$, and $\theta$ a quasi-character of $E^\times$. Let $c = f_E(\theta)$ be the conductorial exponent of $\theta$, that is, $\theta$ is trivial on $1 + P_E^c$ but not on $1 + P_E^{c-1}$. If $c > 1$, then there is a unique element $\gamma_\theta \in C_E$ such that $\theta(1+x) = \psi_E(\gamma_\theta x)$ for $x \in P_E^{c-1}$. $\gamma_\theta$ is called the standard representative of $\theta$, and we say that $\gamma_\theta$ represents $\theta$. Note that $\text{val}_E(\gamma_\theta) = 1 - c$, and if $K/E$ is a finite extension and $\phi = \theta \circ N_{K/E}$, then $\gamma_\phi = \gamma_\theta$.

**Definition 2.3** (Kutzko, \cite{Ku80}). Let $E/F$ be an extension of degree $N$, $p$ doesn’t divide $N$. A quasi-character $\theta$ of $E^\times$ is generic over $F$ if either

(i) $f_E(\theta) = 1$, $E/F$ is unramified and $\theta$ doesn’t come via the norm from a proper subfield of $E$ containing $F$, or

(ii) $f_E(\theta) > 1$, and $F(\gamma_\theta) = E$.

Note that a generic character is necessarily admissible. If $F \subset L \subset E$ and $\theta$ is a quasi-character of $L^\times$, let $\theta_E = \theta \circ N_{E/L}$, or simply $\theta$ when the context is clear. Viewed as a character of $W_L$, $\theta_E$ is the restriction of $\theta$ to the subgroup $W_E$.

**Lemma 2.4** (Howe’s Factorization Lemma, \cite{Ho77} and \cite{M86}). Let $E/F$ be an extension of degree $N$, $p$ doesn’t divide $N$. Let $\theta$ be an admissible character of $E/F$. Then
there is a unique tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_s = E$ and quasi-characters $\chi = \phi_0, \phi_1, \ldots, \phi_s$ of $F^\times = F_0^\times, F_1^\times, \ldots, F_s^\times = E^\times$ respectively, with $\phi_k$ generic over $F_{k-1}$, $k = 1, 2, \ldots, s$, such that

$$\theta = \chi \phi_1 \phi_2 \cdots \phi_s.$$ 

$\phi_k$'s are well-defined modulo a character coming from $F_{k-1}^\times$ via the norm map, and the conductoral exponents $f_E(\phi_1) > f_E(\phi_2) > \cdots > f_E(\phi_s)$ are unique.

In [Ho77], Howe constructed supercuspidal representations of $G_N(F)$ from admissible characters of $E^\times$ with respect to $F$ when $E/F$ is tamely ramified of degree $N$. In [M86], Moy showed that Howe’s construction exhausted all the supercuspidal representations in this case. Given an admissible character of $E^\times$, denote by $\pi_\theta$ the supercuspidal representation constructed by Howe.

In [M86], Moy also computed the $\epsilon$-factors of $\theta$ as follows. Let $\theta$ a quasi-character of $F^\times$. Let $c = f(\theta)$ be the conductoral exponent of $\theta$, that is, $\theta$ is trivial on $1 + p^c_F$ but not on $1 + p^{c-1}_F$.

When $f(\theta) > 1$, let $r = \lceil f(\theta)/2 \rceil$. Define $c_\theta \in p^{-f(\theta)}_F \mod p^{-r-1}_F$ such that

$$\theta(1 + x) = \psi(c_\theta x),$$

for $x \in p^r_F$. If $E/F$ is a finite extension, then $c_{\theta \circ \sqrt[p]{N}} = c_\theta$.

Suppose $f(\theta) = f_F(\theta) = 2n + 1$ is odd. Set $H = U^n_F, J = U^{n+1}_F$. We define the Gauss sum

$$G(\theta, \psi) = q^{-\frac{1}{2}} \sum_{x \in H/J} \theta^{-1}(x) \psi_F(c_\theta(x - 1)).$$

Denote by $\epsilon(\theta, \psi)$ the value of the $\epsilon$-factor at $s = 0$. Note that $\epsilon(s, \theta, \psi) = \epsilon(0, \theta, |\cdot|^s, \psi)$.

**Theorem 2.5** (Moy, [M86], (2.3.17)).

(i) Suppose $f(\theta)$ is even. Then

$$\epsilon(\theta, \psi) = \theta^{-1}(c_\theta) \psi(c_\theta) |c_\theta|^{1/2}.$$  

(ii) Suppose $f(\theta)$ is odd. Then

$$\epsilon(\theta, \psi) = \theta^{-1}(c_\theta) \psi(c_\theta) |c_\theta|^{1/2} G(\theta, \psi).$$

3. Construction

In this section, we explicitly construct two non-isomorphic supercuspidal representations of $G_N(F)$ which will later show that the bound $\lceil N^2 \rceil$ of $r$ in Conjecture 1.2 is sharp. From now until the end of the paper, we assume that $p > N$.

Fix a uniformizer $\varpi_F$ in $F$, and define a totally ramified extension $E = F[\sqrt[p]{\varpi_F}]$. Since we assume that $p > N$, in particular, $p \nmid N$, $E$ is a tamely ramified extension of $F$ of degree $N$. Set $\varpi_E := \sqrt[p]{\varpi_F}$. 
Recall that we have assumed that $N \geq 5$. The element $\beta := \varpi_E^{2-2N}$ defines a character of $1 + \mathcal{P}_E^{2N-2}$, trivial on $1 + \mathcal{P}_E^{2N-1}$. When $N$ is odd, we extend this character in any way to $E^\times$, and denote the resulting extension by $\phi$. That $\phi$ is admissible follows readily since one can see that it is generic (see Definition 2.3).

When $N$ is even, the element $\beta_1 = \beta = \varpi_E^{2-2N}$ is in $C_{E_1}$, where $E_1 = F[\beta_1] \subsetneq E$. We use the same construction as in the odd case to obtain a character $\phi_1$ on $E_1^\times$. We then choose an integer $\ell \in [2, N - 1]$ that is coprime to $N$ and define $\beta_2 = \varpi_E^{-\ell}$. This condition implies that $E = E_1[\beta_2]$, and $\beta_2 \in C_E$. Similar to above, we define a character $\phi_2$ on $1 + \mathcal{P}_E$ trivial on $1 + \mathcal{P}_E^{\ell+1}$ by $\beta_2$, and extend it to a character, still denoted by $\phi_2$, on $E^\times$. Now take

$$\phi = (\phi_1 \circ N_{E/E_1})\phi_2.$$ 

The above constructions imply that $\phi$ is admissible over $F$, and that the product above is the Howe factorization of $\phi$. We emphasize that we have now defined a character $\phi$, which has a different definition when $N$ is odd from when $N$ is even.

We now define two characters $\phi^{(1)}, \phi^{(2)}$, on $E^\times$ by first setting $\phi^{(1)}(\varpi_E) = \phi^{(2)}(\varpi_E)$ and $\phi^{(1)}|_{k_E^\times} = \phi^{(2)}|_{k_E^\times}$. We then set $\phi^{(2)}|_{1+\mathcal{P}_E^2} = \phi^{(1)}|_{1+\mathcal{P}_E^2}$, and then extend $\phi^{(i)}|_{1+\mathcal{P}_E^2}$ to $1 + \mathcal{P}_E$ in two different ways to produce two characters $\phi^{(1)}, \phi^{(2)}$, that differ on $(1 + \mathcal{P}_E) \setminus (1 + \mathcal{P}_E^2)$.

We observe that $\phi^{(1)}$ and $\phi^{(2)}$ can be chosen in a way as to not be isomorphic as admissible pairs. Indeed, there are $q - 1$ nontrivial characters of $1 + \mathcal{P}_E$ that are trivial on $1 + \mathcal{P}_E^2$. However, there are at most $[E : F]$ admissible characters of $E^\times$ that are conjugate to any given admissible character of $E^\times$ (since two characters of $E^\times$ are isomorphic if and only if there exists an $F$-automorphism of $E$ taking one character to the other). But $[E : F] = N$, and we have assumed from the beginning that $p > N$.

**Remark 3.1.** The requirements that we have placed on $\phi^{(1)}, \phi^{(2)}$ imply $\phi^{(1)}|_{F^\times} \equiv \phi^{(2)}|_{F^\times}$ since $\phi^{(1)}(\varpi_E) = \phi^{(2)}(\varpi_E)$, $\phi^{(1)}|_{k_E^\times} \equiv \phi^{(2)}|_{k_E^\times}$, and $1 + \mathcal{P}_F \subset 1 + \mathcal{P}_E^2$. We will need this later.

Set $\pi_i = \pi_{\phi^{(i)}}$. We see by the above construction that $\pi_1 \not\cong \pi_2$. The following is our main result, showing that the bound $[N/2]$ of $r$ in Conjecture 1.2 is indeed sharp.

**Theorem 3.2.**

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable $s$, for all irreducible supercuspidal representations $\tau$ of $G_r$ with $r = 1, \ldots, [N/2] - 1$. 


4. Proof of Theorem 3.2

Henceforth, if $T/F$ is an extension, we denote by Gal$(T/F)$ the set of all embeddings of $T$ into $F$ that fix $F$, whether or not $T/F$ is Galois. Moreover, for a character $\chi$ of $F^\times$, we denote by $\chi_{T/F}$ the character $\chi \circ N_{T/F}$.

**Proof of Theorem 3.2**. Let $\tau$ be any irreducible supercuspidal representation of $G_r$, $1 \leq r \leq [\frac{N}{2}] - 1$. Since $p \nmid r$, by the discussion in Section 2, one may assume that the local Langlands parameter of $\tau$ is Ind$_W^F F \lambda$, for some admissible pair $(L/F, \lambda)$.

Let $f_L(\lambda)$ be the conductoral exponent of $\lambda$, and let $m = f_L(\lambda) - 1$, so that there exists an element $\alpha \in \mathcal{P}^{-m} \mod \mathcal{P}^{-[\frac{m}{2}]}$ that represents $\lambda$, that is,

$$\lambda(1 + x) = \psi_L(\alpha x),$$

for $x \in \mathcal{P}^{-[\frac{m}{2}]} + 1$. Let $e_L = e(L/F)$ be the ramification index of $L/F$. Finally, let $M$ be the Galois closure of $K = EL$, over $F$.

To prove (3.1), it suffices by the local Langlands correspondence for $GL_N$ to prove that

$$\gamma(s, \text{Ind}_{W_L}^F \phi^{(1)} \otimes \text{Ind}_{W_L}^F \lambda, \psi) = \gamma(s, \text{Ind}_{W_L}^F \phi^{(2)} \otimes \text{Ind}_{W_L}^F \lambda, \psi)$$

for all such $(L/F, \lambda)$.

Let $\theta_g^{(i)} = [g \phi^{(i)} \circ N_{L(9E)/9E}] \otimes (\lambda \circ N_{L(9E)/L})$, for $g \in W_L \setminus W_F/W_E$, and write $K_g = L(9E)$, $E_g = 9E$. We note that $9\beta + \alpha$ represents $\theta_g^{(i)}$, for both $i = 1, 2$.

By Lemma 4.1, 4.2, 4.3, one has to show that

$$\prod_{g \in W_L \setminus W_F/W_E} g \phi^{(1)} \circ N_{K_g/E_g} (9\beta + \alpha) = \prod_{g \in W_L \setminus W_F/W_E} g \phi^{(2)} \circ N_{K_g/E_g} (9\beta + \alpha)$$

We separate the proof of (4.2) into three cases: (1) $\text{val}_K(\beta) = \text{val}_K(\alpha)$; (2) $\text{val}_K(\beta) < \text{val}_K(\alpha)$; (3) $\text{val}_K(\beta) > \text{val}_K(\alpha)$, and we note that $\text{val}_{K_g}(9\beta) = \text{val}_K(\beta)$, $\text{val}_{K_g}(\alpha) = \text{val}_K(\alpha)$ $\forall g \in W_L \setminus W_F/W_E$. Let $e = e(K/E)$ and $N' = e(K/L)$. We have a diagram of fields, with ramification indices listed on the line segments.
We note in particular that since $E/F$ is totally ramified, we have that $N = e_2e'_2$. We also have $\beta \in \mathcal{P}_K^{-e(2N-2)}$ and $\alpha \in \mathcal{P}_K^{mN'}$.

**Case (1).** $\text{val}_K(\beta) = \text{val}_K(\alpha)$. Then, $e(2N - 2) = mN'$. Since $E/F$ is totally ramified, and by the diamond, we have that $N' = \frac{e_2}{e_1}$. Altogether, the equality $e(2N - 2) = mN'$ can be rewritten now as

$$2e_2e'_2 - 2 = m \frac{e_2}{e_1}.$$  

Multiplying both sides of the equality by $e_1e'_2$ yields

$$2N e_1e'_2 - 2e_1e'_2 = mN.$$  

Simplifying, we get $N(2e_1e'_2 - m) = 2e_1e'_2 - 2e_L$. Therefore, we obtain that $N$ divides $2e_L$. On the other hand, we have $1 \leq e_L \leq r < \frac{N-1}{2}$, a contradiction.

**Case (2).** $\text{val}_K(\beta) < \text{val}_K(\alpha)$. Since $\varphi^{(1)}_{K_\alpha/E_\alpha} (\alpha) = \varphi^{(2)}_{K_\alpha/E_\alpha} (\beta)$, we just have to show that

$$\prod_{g \in W_L \setminus W_F/W_E} g \varphi^{(1)}_{K_\alpha/E_\alpha} (1 + (\beta)^{-1}) = \prod_{g \in W_L \setminus W_F/W_E} g \varphi^{(2)}_{K_\alpha/E_\alpha} (1 + (\beta)^{-1})$$

We compute the left hand side.

$$\prod_{g \in W_L \setminus W_F/W_E} g \varphi^{(1)}_{K_\alpha/E_\alpha} (1 + (\beta)^{-1})$$

$$= \varphi^{(1)} \left( \prod_{g \in W_L \setminus W_F/W_E} \left( g^{-1} \left( \prod_{\sigma \in \text{Gal}(K_\alpha/E_\alpha)} (1 + \sigma((\beta)^{-1})) \right) \right) \right)$$

$$= \varphi^{(1)} \left( \prod_{g \in W_L \setminus W_F/W_E} \prod_{\sigma \in \text{Gal}(K_\alpha/E_\alpha)} g^{-1}(1 + \beta^{-1}\sigma(\alpha)) \right)$$

$$= \varphi^{(1)} \left( \prod_{g \in W_L \setminus W_F/W_E} \left( \prod_{\sigma \in \text{Gal}(K_\alpha/E_\alpha)} (1 + \beta^{-1}(g^{-1}\sigma(\alpha))) \right) \right)$$

since $\beta^{-1} \in E_\alpha$ is fixed by all $\sigma$.

We now note that

$$W_L \setminus W_F/W_E \cong W_E \setminus W_F/W_L \cong \text{Gal}(M/E) \setminus \text{Gal}(M/F)/\text{Gal}(M/L),$$

where $M$ is the Galois closure of the compositum $EL$ (see [ALST18, Lemma 2.6]. We obtain that $W_L \setminus W_F/W_E$ can be interpreted as the orbits in $\text{Gal}(M/F)/\text{Gal}(M/L) = \text{Gal}(L/F)$, under the action of $\text{Gal}(M/E)$. To view this action, we can think of $\text{Gal}(L/F)$ as embeddings of $L$ into $M$ (since $M$ is the Galois closure of $EL$) that fix $F$.  

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We are therefore considering a set of elements $g^{-1}\sigma$, where $g$ ranges over a set of representatives of the orbits of $\text{Gal}(M/E)$ acting on $\text{Gal}(L/F)$, and $\sigma \in \text{Gal}(K_g/E_g)$. We wish to show that this set of $g^{-1}\sigma$ exhausts all of $\text{Gal}(L/F)$. In fact it is better to write $g^{-1}\sigma$ as $(g^{-1}\sigma g)g^{-1}$. We will show that these elements exhaust all of $\text{Gal}(L/F)$.

So we now prove that

$$\{g^{-1}\sigma : g \in \text{Gal}(M/E)\setminus\text{Gal}(M/F)\}/\text{Gal}(M/L), \sigma \in \text{Gal}(K_g/E_g)\}$$

can be identified canonically with $\text{Gal}(L/F)$, which would simplify the above expression to

$$\phi^{(1)} \left( \prod_{\sigma \in \text{Gal}(L/F)} (1 + \beta^{-1}\sigma(\alpha)) \right).$$

Let $H = \text{Gal}(M/E), G = \text{Gal}(M/F), J = \text{Gal}(M/L)$, so that $H$ acts on $G/J$. We consider an orbit $Hg^{-1}J, g \in G$. By the orbit-stabilizer theorem, $Hg^{-1}J \cong \text{Stab}_{H}(g^{-1}J)$. Furthermore, we argue that $\text{Stab}_{H}(g^{-1}J) \cong \text{Gal}(M/(g^{-1}K_g))$. Indeed, first note that $x \in \text{Gal}(M/E)$ stabilizes $g^{-1}J$ if and only if $xg^{-1}J = g^{-1}J$, i.e. that $x \in g^{-1}Jg = \text{Gal}(g^{-1}M/g^{-1}L) = \text{Gal}(M/g^{-1}L)$, since $g^{-1}M = M$. Therefore,

$$x \in \text{Stab}_{H}(g^{-1}J)$$
$$\iff x \in \text{Gal}(M/E) \cap \text{Gal}(M/(g^{-1}L))$$
$$\iff x \in \text{Gal}(M/((g^{-1}L)E) = \text{Gal}(M/(g^{-1}K_g))$$

We conclude that $Hg^{-1}J \cong \text{Stab}_{H}(g^{-1}J) \cong \frac{\text{Gal}(M/E)}{\text{Gal}(M/(g^{-1}K_g))} \cong \text{Gal}(g^{-1}K_g/E)$, noting that $g^{-1}K_g = (g^{-1}L)E$. Now, via $\eta \mapsto g\eta g^{-1}$, we have $\text{Gal}(g^{-1}K_g/E) \cong \text{Gal}(K_g/E_g)$. The identifications above now allow us to conclude an isomorphism $\text{Gal}(K_g/E_g) \cong \text{Stab}_{H}(g^{-1}J)$ given by $\sigma$ maps to the class of $g^{-1}\sigma g$, with $\sigma \in \text{Gal}(K_g/E_g)$. Therefore, the orbit $Hg^{-1}J$ can be identified with the set of $(g^{-1}\sigma g)g^{-1}J$ such that $\sigma \in \text{Gal}(K_g/E_g)$, i.e. the set of $g^{-1}\sigma J$, which is exactly what we set out to prove.

So we obtain

$$\phi^{(1)} \left( \prod_{\sigma \in \text{Gal}(L/F)} (1 + \beta^{-1}\sigma(\alpha)) \right).$$

We now have

$$\phi^{(1)} \left( \prod_{\sigma \in \text{Gal}(L/F)} (1 + \beta^{-1}\sigma(\alpha)) \right) = \phi^{(1)} \left(1 + \sum_{i=1}^{s} \beta^{-1} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha)\right)$$

where $s = \#\text{Gal}(L/F)$, and where $B_i$ is the set of all $i$-element subsets of $\text{Gal}(L/F)$. 
Claim: \( P(\alpha) := \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \in F \). To see this, firstly, the sum is over \( Gal(L/F) \).

By construction, \( P(\alpha) \) is fixed by every embedding \( h \) in \( Gal(L/F) \) because composition with \( h \) just permutes the embeddings in \( Gal(L/F) \), so \( h \) just permutes the terms in the sum giving \( P(\alpha) \).

Since \( \beta^{-1} \in \mathcal{P}_E^{2N-2} \), and \( \alpha \in \mathcal{P}_{L^{-m}} \), we have

\[
\beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \in \mathcal{P}_E^{i(2N-2)+e_2' e_2' \left\lceil \frac{-im}{e_1' e_2'} \right\rceil},
\]

noting that \( \mathcal{P}_L^{-im} \cap F = \mathcal{P}_F^{\left\lceil \frac{-im}{e_1' e_2'} \right\rceil} \) (this is where we use the above claim that \( P(\alpha) \in F \)).

We write

\[
A := i(2N - 2) + e_2' e_2' \left\lceil \frac{-im}{e_1' e_2'} \right\rceil = i(2N - 2) + N \left\lceil \frac{-im}{e_1' e_2'} \right\rceil
\]

Modulo \( N \), we get

\[
A \equiv -2i \pmod{N},
\]

Since \( 1 \leq i \leq s \), we have that, modulo \( N \), \( 2 \leq -A \leq 2s \). Note also that \( 1 \leq r < \frac{N-1}{2} \). Therefore, we have that, modulo \( N \), \( 2 \leq -A \leq 2s \leq 2r < N - 1 \). In particular, one now sees that \( A \neq 1 \mod N \). In particular, \( A \neq 1 \). Hence, \( A \geq 2 \), and \( 1 + \sum_{i=1}^{s} \beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \in 1 + \mathcal{P}_E^2 \). Therefore, since \( \phi^{(1)}, \phi^{(2)} \) agree on \( 1 + \mathcal{P}_E^2 \), we have

\[
\phi^{(1)} \left( 1 + \sum_{i=1}^{s} \beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \right) = \phi^{(2)} \left( 1 + \sum_{i=1}^{s} \beta^{-i} \sum_{H \in B_i} \prod_{\sigma \in H} \sigma(\alpha) \right),
\]

completing the proof of Case (2).
Case (3). \( val_K(\beta) > val_K(\alpha) \). This proof here is similar to the proof of case (2). We note that

\[
\prod_{g \in W_L \setminus W_F / W_E} g \phi(i) \circ N_{K_g / E_g}(\alpha)
\]

\[
= \phi(i) \left( \prod_{g \in \text{Gal}(M/E) \setminus \text{Gal}(L/F)} \prod_{\sigma \in \text{Gal}(K/E)} \sigma(\alpha) \right)
\]

\[
= \phi(i) \left( \prod_{g \in \text{Gal}(K/E) \setminus \text{Gal}(L/F)} \prod_{\sigma \in \text{Gal}(K/E)} \sigma(\alpha) \right)
\]

\[
= \phi(i) \left( \prod_{\sigma \in \text{Gal}(L/F)} \sigma(\alpha) \right)
\]

\[
= \phi(i) (N_{L/F}(\alpha)),
\]

and since \( \phi(1)|_{F^*} \equiv \phi(2)|_{F^*} \) (see Remark 3.1), we see that (4.2) is equivalent to

\[
\prod_{g \in W_L \setminus W_F / W_E} g \phi(1) \circ N_{K_g / E_g}(1 + g \beta \alpha^{-1}) = \prod_{g \in W_L \setminus W_F / W_E} g \phi(2) \circ N_{K_g / E_g}(1 + g \beta \alpha^{-1})
\]

But the argument now is the same as in case (2). On the left, we arrive at the term

\[
\phi(1) \left( \prod_{\sigma \in \text{Gal}(L/F)} (1 + \beta \sigma(\alpha^{-1})) \right),
\]

and also the term (using the same notation as in case (2)) \( A = -(i(2N - 2) + e_2 e_2 [\frac{\ln \gamma}{\pi i e_2^2}]) \), which is precisely the negative of the term \( A \) that appeared in case (2). We still have \( A > 0 \), since \( val_K(\beta) > val_K(\alpha) \). Then, \( A \equiv 2i \ (\text{mod} \ N) \), and \( 2 \leq A < N - 1 \), so that \( A \neq 1 \) mod \( N \) and thus \( A \neq 1 \). Hence, we have proven Case (3).

This completes the proof of Theorem 3.2 up to the the proofs of Lemmas 4.1, 4.2, and 4.3. □

4.1. Lemmas. In this section, we lay out the various Lemmas that we needed in the proof of Theorem 3.2.

Lemma 4.1. [ALST18, Lemma 2.5] For \( i = 1, 2 \), \( (\text{Ind}_{W_E}^{W_{L,E}} \phi(i)) \otimes (\text{Ind}_{W_L}^{W_{E}} \lambda) \) is isomorphic to

\[
\bigoplus_{g \in W_L \setminus W_F / W_E} \text{Ind}_{W_{L,(gE)}}^{W_{E}} [(g \phi(i) \circ N_{L,E}(gE)) \otimes (\lambda \circ N_{L,E(L/E)})].
\]
Recall that $\theta_g^{(i)} = [(g \phi^{(i)} \circ N_{L(E)/E}) \otimes (\lambda \circ N_{L(E)/L})]$, for $g \in W_L \setminus W_F/W_E$, and $K_g = L(E), E_g = E$. We also write $\nu_g^{(i)} = [(g \phi^{(i)} \circ N_{M_g/E_g}) \otimes (\lambda \circ N_{M_g/L})]$ for $i = 1, 2$.

Since the $L$-function is inductive, multiplicative, and trivial on ramified characters, and by [ALST18, Lemma 2.4], we have

**Lemma 4.2.** We have for $i = 1, 2$,

$$\gamma(s, \bigoplus_{g \in W_L \setminus W_F/W_E} \text{Ind}^W_{W_Kg} \theta_g^{(i)}, \psi_F) = \prod_{g \in W_L \setminus W_F/W_E} \lambda_{K_g/F}(\psi_F) \gamma(s, \theta_g^{(i)}, \psi_{K_g}),$$

where $\lambda_{K_g/F}(\psi_F)$ is the Langlands constant associated to $K_g/F$ and $\psi_F$ (see [BH06, Theorem 29.4], [BH06, §34.3]), and $\psi_{K_g} = \psi_F \circ \text{tr}_{K_g/F}$. The formula is also true when $\gamma$ is replaced by $\epsilon$.

**Lemma 4.3.**

$$\frac{\gamma(s, \theta_g^{(1)}, \psi_{K_g})}{\gamma(s, \theta_g^{(2)}, \psi_{K_g})} = \frac{\theta_g^{(1)}(g \beta + \alpha)}{\theta_g^{(2)}(g \beta + \alpha)}$$

**Proof.** It suffices to prove the formula with $\gamma$ replaced by $\epsilon$, by the proof of Lemma 4.2.

We first note that $c_{g^{(1)}} = c_{g^{(2)}} = g \beta + \alpha$ so that in particular $f := f_{K_g}(\nu_g^{(1)}) = f_{K_g}(\nu_g^{(2)})$. If $f$ is even, then the the proposed equality is easily seen to be true by Theorem 2.5.

Suppose that $f = 2n + 1$ is odd. We must check that $G(\theta_g^{(1)}, \psi_{K_g}) = G(\theta_g^{(2)}, \psi_{K_g})$. The impact of $g \in W_L \setminus W_F/W_E$ is negligible, so we will assume that $g = 1$.

Set $\theta^{(i)} = \phi^{(i)} \circ N_{LE/E} \otimes \lambda \circ N_{LE/L}$, $K = EL$, and $f = f_K(\theta^{(1)}) = f_K(\theta^{(2)}) = 2n + 1$ is odd. Since $c_{g^{(1)}} = c_{g^{(2)}} = \beta + \alpha$, and by definition of $\theta^{(i)}$, it suffices to check that

$$\phi^{(1)} \circ N_{K/E}(x) = \phi^{(2)} \circ N_{K/E}(x) \text{ for } x \in U_K^n = 1 + P_K^n.$$  

We first note that $f = \max\{e(2N - 2), mN\} + 1$. Therefore, since $N \geq 5$, we have that $n \geq \frac{3}{2} > e$, so that $N_{K/E}(U_K^n) \subset U_E^2$ by a standard property of the norm map. Since $\phi^{(1)}$ and $\phi^{(2)}$ agree on $U_E^2$, we have that (4.4) is proven. 

**REFERENCES**


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