

02/25/15

Xin Shen - Fourier Coefficients of Automorphic Representations

- 1. Constructing ~~of~~ automorphic ~~representations~~ L-functions
 - Rankin Selberg
 - Langlands - Shahidi

- 2. Construction of functorial liftings
 - Descent

Fourier
Coefficients

Whittaker Coefficients

~~$G = GL_n$~~ or ~~$G = GL_N$~~ \mathbb{R}^{2n}
 $G = GL_n$ or Sp_{2n}

$$N = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & * & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

$F =$ number field
 $A = A_F$

$\psi_N : N \rightarrow \mathbb{C}^*$ standard non-degenerate Whittaker character for G .

Let φ be an automorphic form on $G(A)$.

Can define Whittaker coefficient $W_\varphi(g) = \int_{N(F) \backslash N(A)} \varphi(ng) \psi_N(n) dn$

Parameterization of Fourier coefficients

$$\text{let } G = \mathfrak{sp}_{2n}$$

$$\{\text{Fourier coefficients}\} \longleftrightarrow \left\{ \begin{array}{l} \text{unipotent conjugacy} \\ \text{classes in } G(\mathbb{F}) \end{array} \right\}$$

Ex: $\mathbb{Q}_r \overset{\text{In}}{\subset} G(\mathbb{F})$, the unipotent classes are parameterized by "symplectic partitions of the integer $2n$ " i.e. every odd number occurs an even number of times.

for example, in \mathfrak{sp}_6 , $[4\ 2]$ and $[3\ 3]$ works, but $[5\ 1]$ doesn't.

$\theta = [4\ 2]$: Get one-dimensional torus

$$T_{\theta}(t) = \begin{pmatrix} t^3 & & & & & \\ & t & & & & \\ & & t & & & \\ & & & t^{-1} & & \\ & & & & t^{-1} & \\ & & & & & t^{-3} \end{pmatrix}$$

Then $T_{\theta} \hookrightarrow \sigma$, giving

$$\sigma = \bigoplus \sigma_i$$

$$\sigma_i = \{ X \mid \text{Ad}(T_{\theta}(t)X) = t^i \cdot X \}$$

Eigenvalues for σ_i :

$$\begin{pmatrix} t^2 & 2 & 4 & 6 \\ t & 2 & 4 & \\ t^{-1} & & 2 & \\ & t^{-1} & & t^{-3} \end{pmatrix}$$

$$L_{\mathcal{O}} = \exp(\sigma_{\mathcal{O}})$$

$L_{\mathcal{O}}(\bar{F})$ acts on σ_2 with an open

orbit. A ~~representative~~ representative of this orbit is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Get unipotent element from \nearrow

$$N = \exp\left(\sum_{i=1,2} \sigma_i\right)$$

$$\psi_N(n) = \psi(n_{12} + a n_{25} + b n_{34})$$

for some $a, b \in \mathbb{F}^x / (\mathbb{F}^x)^2$

(3)

Define Fourier coefficient

$$f_1(\rho, s) = \int_{N(F) \backslash N(\mathbb{A})} \psi(\rho g) \psi_N(n) dn$$

Partial order on partitions of \mathbb{Z}^n :

$$\sigma = (a_1, a_2, \dots, a_k) \quad a_1 \geq a_2 \geq \dots \geq a_k$$

$$\sigma' = (b_1, b_2, \dots, b_l) \quad b_1 \geq b_2 \geq \dots \geq b_l$$

$$\sigma \geq \sigma' \quad \text{if} \quad \sum_{i=1}^N a_i \geq \sum_{i=1}^N b_i \quad \forall N.$$

$$\sigma \geq \sigma' \iff \bar{\sigma} \supset \bar{\sigma}'$$

Let π be an automorphic representation.

$$\mathcal{O}_F(\pi) = \{ \text{max } \sigma \mid \pi \text{ has nonzero} \}$$

Fourier coefficient associated to σ

Q: Given σ , can we find π such that $\sigma \in \mathcal{O}_F(\pi)$?

"wave front set"

Assume that π is cuspidal. Then:

① For $G = GL_n$, cuspidal \Rightarrow generic.

Thus, $\mathcal{O}_G(\pi) = \{[n]\}$

② For ~~power~~ F -adic, the result is due to Mœglin-Waldspurger.

③ For $G = Sp_{2n}$, Ginzburg/Rallis/Soudry conjecture ~~that~~ if π is cuspidal, then

$\mathcal{O}_G(\pi)$ consists of even partitions.

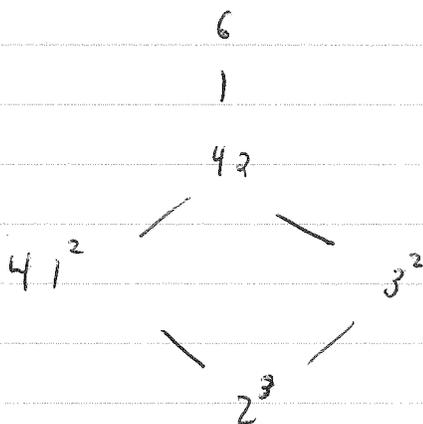
④ Jian-shu Li: $\mathcal{O}_G(\tilde{\pi})$ has nontrivial Fourier

coefficient associated to $[2^n]$, if $\tilde{\pi}$ is cuspidal.

Theorem (-): ③ is true.

Ex for Sp_6 :

Partitions structure:



Need to show

that if 3^2 ~~occurs~~ occurs,

then it has a nontrivial Fourier coefficient associated to $[4, 2]$ or $[6]$.

⑤