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Cheng-Chung Tsai - Orbital Integrals and character expansion of supercuspidal representations

$F = p\text{-adic field}$ ,  $E = F(\sqrt{a})$  ramified quadratic extension.

$\mathcal{B}$  = residue field, with  $\zeta_p \in \text{clark}(\mathcal{B}) > 0$

$V = E^n$  Hermitian space, with form

$$\langle x, y \rangle = x_1 y_1^* + x_2 y_2^* + \dots + x_n y_n^*$$

where  $y_i^* = \sigma(y_i)$ ,  $\sigma \in G(\mathbb{A}/E/F)$

$\#$  denotes  $\underline{\mathcal{U}}$

This data gives a quasisplit unitary group over  $F$ , split over  $E$ ,

such that  $\underline{\mathcal{U}}(F) = U(V) = \text{unitary operators on } V$

Write  $G := \underline{\mathcal{U}}(F)$ ,  $\mathfrak{g} = \text{Lie}(G) = \left\{ \begin{array}{l} \text{anti-Hermitian} \\ \text{endomorphisms} \end{array} \right\} \text{ of } V$

Let  $\Lambda = \mathcal{O}_E^\times \subset E^\times = V$  ✓ building

$\Lambda$  gives you a point  $x \in \mathcal{B}(\underline{\mathcal{U}}, F)$

with  $G_x = U(\Lambda)$ ,  $\mathfrak{g}_x = \left\{ \begin{array}{l} \text{anti-Hermitian endomorphisms} \\ \text{on } \Lambda \end{array} \right\}$

Have Moy-Prasad filtration  $G_x \supset G_{x,1/2} \supset G_{x,1} \supset \dots$

$\dots \supset \mathfrak{g}_{x,1} \supset \mathfrak{g}_{x,1/2} \supset \mathfrak{g}_x \supset \mathfrak{g}_{x,1/2} \supset \mathfrak{g}_{x,1} \supset \dots$

$$G_{x, \frac{d}{2}} = \ker \left( G_x \longrightarrow \mathcal{O} \left( \frac{1}{\pi^{\frac{d}{2}} \Lambda} \right) \right)$$

$$\mathcal{O}_{x, \frac{d}{2}} = \left\{ x \in \mathcal{O} : x(\Lambda) \subset \pi^{\frac{d}{2}} \Lambda \right\}$$

Let  $\bar{V} := \Lambda / \pi^{\frac{d}{2}} \Lambda$  = n-dim'l quadratic space over  $\mathbb{K}$   
with  $(x, y) = x_1 y_1 + \dots + x_n y_n$ ,

$$\mathcal{O}_{x, m} / \mathcal{O}_{x, m+\frac{1}{2}} \xrightarrow{\sim \cdot \pi^{-m}} \mathfrak{SO}(\bar{V}) \quad \text{for } m \in \mathbb{Z}$$

$$\mathcal{O}_{x, m+\frac{1}{2}} / \mathcal{O}_{x, m+1} \xrightarrow{\sim \cdot \pi^{-m-\frac{1}{2}}} \mathrm{End}^{G_m}(\bar{V})$$

$$G_x / G_{x, \frac{d}{2}} \xrightarrow{\sim} O(\bar{V})$$

let  $\tilde{\pi}$  be an irreducible admissible repn of  $G$ .

Then: (Harish-Chandra)

$$Q_{\tilde{\pi}}(f \circ \log) = \sum_{\theta \in O(\mathfrak{o})} c_{\theta}(\tilde{\pi}) \hat{\mu}_{\theta}(f) \quad \forall f \in C_c^\infty(V_{\tilde{\pi}})$$

where  $\theta \in V_\eta \subset \mathfrak{g}$  is sufficiently small and  $\mathcal{O}$

and  $\mathcal{O}(\theta) = \text{set of nilpotent orbits in } \mathfrak{g}$ .

Thm (Shalika/Waldspurger/DeBacker):

$\forall T \in \mathfrak{g}, \exists \text{ lattice } \Lambda_T \subset \mathfrak{g} \text{ such that}$

$$\mu_T(t) = \sum_{\theta \in \mathcal{O}(p)} P_\theta(T) \mu_\theta(t)$$

$$\forall f \in C_c^\infty(\mathfrak{g}/\Lambda_T)$$

$P_\theta(T)$  are called Shalika germs.

Fix additive character  $\psi: F \rightarrow \mathbb{C}^*$  such that  $\psi|_{F^\times} = 1$ ,

$$\psi|_{\mathfrak{o}} \neq 1$$

let  $\text{Tr}: \mathfrak{g} \times \mathfrak{g} \rightarrow F$  &  $\phi$  the trace perfect pairing.

$$\text{Def } \psi \circ \text{Tr} : \mathfrak{J}_{x,d} / \mathfrak{J}_{x,\frac{d}{2}} \times \mathfrak{J}_{x,-d} / \mathfrak{J}_{x,-d+\frac{1}{2}} \rightarrow \mathbb{C}^*,$$

a perfect pairing.

$$\text{Fix } d \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}. \text{ let } T \in \text{End}^{S_{\text{fin}}}(\bar{V}) \cong \mathfrak{J}_{x,-d} / \mathfrak{J}_{x,-d+\frac{1}{2}}$$

$T$  gives (by the above perfect pairing) a function

$$\psi_T \text{ on } \mathcal{O}_{x,d} / \mathcal{O}_{x,d+\frac{1}{2}} \xrightarrow{\sim} G_{x,d} / G_{x,d+\frac{1}{2}}$$

unreduced component exponential map

Assume that the stabilizer of  $T$  in  $G / G_{x,d+\frac{1}{2}}$  is anisotropic.

Fact:  $\text{c-Ind}_{G_{x,d}}^G \psi_T$  is a finite direct sum of irreducible supercuspidal representations.

let  $\tilde{T}$  be a lift of  $T$  to  $\mathcal{O}_{x,-d}$

Thm (Kim - Murnaghan): let  $\pi \subset \text{c-Ind}_{G_{x,d}}^G \psi_T$  be irreducible.

Then  $\theta_\pi = \text{constant} \cdot \widehat{\mu}_{\tilde{T}}$  on  $G_{x,d} \cong \mathcal{O}_{x,d}$

(he said he was being slightly sloppy here)

Cor:  $c_\theta(\pi) = \text{constant} = \int_0^\infty (\tilde{T})$

Now let  $n=3$ ,  $G=U(3)$ . Consider  $\theta = \theta_{\text{sub}} = \text{orbit of}$

$$N_{\text{sub}} = \pi^{-\frac{1}{2}} \begin{pmatrix} 0 & & \\ & 0 & \\ 1 & & 0 \end{pmatrix}$$

$$\text{Define } f_{N_{\text{sub}}, r} = \frac{1}{\pi^{-2r}} (\text{Ad}(G_K)(N_{\text{sub}} + \mathcal{O}_{K,r}))$$

Lemma: (nilpotent orbit  $\bar{\theta}$ )  $\cap \text{supp}(f_{N_{\text{sub}}, r}) \neq \emptyset \iff \bar{\theta} \supseteq \theta_{\text{sub}}$

Want to compute  $P_{\theta}^r(T)$ . Strategy: Cook up some nice test functions like above, compute  $\mu_{\theta}^r(f)$ , and compute  $\mu_T^r(f)$ .

Per do you have a system of equations and you can compute  $P_{\theta}^r(T)$ ?

Now, for  $r > r_0$  for some  $r_0 > 0$ ,

$$\mu_{\tilde{T}}^r(f_{N_{\text{sub}}}) = \sum_{\bar{\theta} \supseteq \theta_{\text{sub}}} \mu_{\theta}^r(f_{N_{\text{sub}}})$$

( $\theta_{\text{sub}}$  = sub regular orbit)

$$= \sum_{\bar{\theta} \supseteq \theta_{\text{sub}}} P_{\theta}^r(\tilde{T}) (q^{r-r_0})^{\dim(\bar{\theta})} \mu_{\theta}^r(f_{N_{\text{sub}}})$$

$$= P_{\theta_{\text{sub}}}^r(\tilde{T}) q^{-r_0 \dim \theta_{\text{sub}}} \mu_{\theta}^r(f_{N_{\text{sub}}}) (\text{exp})(z)^{\dim(\theta_{\text{sub}})}$$

+ higher terms in  $q^r$

$$\text{Now, recall } V(\beta) = \prod_{\alpha > 0} G_\alpha \left( \underbrace{\begin{pmatrix} \pi^\alpha & \\ & -\pi^\alpha \end{pmatrix}}_{=: d_\alpha} \right) G_\alpha$$

$$m_T(f_{N,r}) = \int_G f_{N,r} (\text{Ad}(g) T) dg$$

$$= \sum_{\alpha > 0} \int_{G_\alpha d_\alpha G_\alpha^{-1}} f_{N,r} (\text{Ad}(g) T) dg$$

$$= \sum_{\alpha > 0} \frac{|G_\alpha d_\alpha G_\alpha^{-1}|}{|G_\alpha|} \int_{G_\alpha} f_{N,r} (\text{Ad}(d_\alpha) \text{Ad}(g) T) dg$$

since  $f_{N,r}$  is  $\text{Ad}(G_\alpha)$ -invariant

This last integral ~~on~~<sup>defined to be</sup> comes down to counting points on a

variety over  $k$ . This variety is

$$\left\{ \bar{g} \in O_3(k) : \text{Ad}(\bar{g}) T = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{pmatrix} \right\}$$

$$\text{or } \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ (k^*)^2 & * & * \end{pmatrix} \right\}$$