An Interpretation of the Tame Local Langlands Correspondence for p-adic $PGSp(4)$ from the Perspective of Real Groups

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Abstract

Let F be a p-adic field. In this paper, we continue the work of the first author and give a new realization of the tame local Langlands correspondence for $PGSp(4, F)$ that is analogous to the construction of the local Langlands correspondence for real groups.

1 Introduction

In this paper, we give a new realization of the tame local Langlands correspondence for $PGSp(4, F)$, where F is a non-Archimedean local field of characteristic zero, using character theory and ideas from the theory of real reductive groups. We assume that the residual characteristic of F is sufficiently large (greater than 89) for technical reasons (see Remark 9.10). We continue the program initiated in [6], which gave a new realization of the local Langlands correspondence for $GL(\ell, F)$, ℓ a prime.

There has been a significant amount of progress in the local Langlands correspondence in recent years. In the recent work of [8], [11], [14], [15], [17], the strategy of constructing a local Langlands correspondence is to first attach a character of a torus to a Langlands parameter, and then to construct a putative L-packet associated to this character.

We propose a different strategy to realize a local Langlands correspondence. Langlands parameters and supercuspidal representations will be parameterized not by characters of elliptic tori, but by characters of finite covers of elliptic tori. Given a supercuspidal Langlands parameter ϕ, we first use a construction of Benedict Gross that naturally associates to ϕ a character ξ of a cover of a subgroup of an elliptic torus. In the case of $PGSp(4, F)$, it is a character of a cover of an elliptic torus. Using character theory, we then attach to ξ a collection of supercuspidal representations of $PGSp(4, F)$. We prove that this collection is the L-packet that is associated to ϕ by DeBacker and Reeder [11]. To do this, we rewrite supercuspidal characters in terms of functions on covers of elliptic tori as in Harish-Chandra's relative discrete series character formula. We then compare our character formulas to those in [11].

Let us briefly recall the construction of [11]. Suppose G is an unramified connected reductive algebraic group defined over F. To a certain class of Langlands parameters (tame, regular, semisimple, elliptic, Langlands parameters, or TRSELPs for short), DeBacker and Reeder associate a character of a torus $T(F)$, to which they attach a collection of supercuspidal representations on the pure inner forms of $G(F)$, a conjectural L-packet. This construction extensively uses Bruhat-Tits theory. They are also able to isolate the part of this L-packet that corresponds to a particular pure inner form, and prove that their correspondence satisfies various natural conditions such as stability of the appropriate sums of characters of representations in the L-packet.

In the theory of real groups, an admissible homomorphism $\mathcal{W}_{\mathbb{R}} \to {}^L G$ for a group $G(\mathbb{R})$ factors through the normalizer of a torus in ${}^L G$. As such, it naturally produce a character $\tilde{\chi}$ of $T(\mathbb{R})_\rho$, where $T(\mathbb{R})_\rho$ is a double cover of some torus $T(\mathbb{R})$. $T(\mathbb{R})_\rho$ is called the *ρ-cover* of $T(\mathbb{R})$. Suppose that $G(\mathbb{R})$ admits relative discrete series representations. Harish-Chandra has calculated the characters of such representations, and these characters are naturally written in terms of genuine functions on $T(\mathbb{R})_\rho$ (see Theorem 3.5). The local Langlands correspondence for these representations of $G(\mathbb{R})$ is then given by attaching to a Weyl group orbit of such characters to $\tilde{\chi}$, and this is the motivation for our work over p-adic fields.

Very recently, Benedict Gross has shown that if one considers a Langlands parameter for a p-adic group that factors through the normalizer of a maximal torus in ${}^L G$, then one obtains something close to a character of a cover of a maximal torus, as follows. Suppose that G is a connected reductive group defined over a p-adic field F. Let $\phi : \mathcal{W}_F \to {}^L G$ be a Langlands parameter for $G(F)$, and suppose that ϕ factors through the normalizer of a maximal torus $\hat{T}$. To ϕ, one can associate a maximal F-torus T in G, unique up to stable conjugacy, whose complex dual is isomorphic to $\hat{T}$. Suppose that T splits over the extension E of F. Then E is necessarily Galois over F, and we set $\Gamma = \mathrm{Gal}(E/F)$. By the local Langlands correspondence for tori, once one chooses a particular isomorphism implementing the duality between T and $\hat{T}$, one can canonically associate to ϕ a character ξ of $T(E)_\Gamma$, the group of coinvariants of $T(E)$ with respect to Γ (see §4).

Invariants and coinvariants are related by the norm map

$$N : T(E) \to T(F)$$
$$t \mapsto \prod_{\xi \in \Gamma} \xi(t)$$

in the cohomology sequence

$$1 \to \hat{H}^{-1}(\Gamma, T(E)) \to T(E)_\Gamma \xrightarrow{N} T(F) = T(E)^\Gamma \to \hat{H}^0(\Gamma, T(E)) \to 1,$$

where $\hat{H}$ denotes Tate cohomology.

Suppose $\hat{H}^0(\Gamma, T(E)) = 0$, in which case $T(E)_\Gamma$ is then a cover of $T(F)$. Let us also suppose that ϕ does not factor through a proper Levi subgroup, so that the representations in the L-packet associated to ϕ are conjecturally all supercuspidal (see [11, §3.5]). We wish to attach a conjectural supercuspidal character to ξ (where ξ is as above), in analogy to the case of real groups. This character will be a p-adic analogue of the Harish-Chandra relative discrete series character formula. Let Δ^+ be a set of positive roots of $T(\overline{F})$ in $G(\overline{F})$. Let ρ denote half the sum of positive roots. Assume now that E/F is unramified. Define

$$\Delta^0(\gamma, \Delta^+) := \prod_{\alpha \in \Delta^+} (1 - \alpha^{-1}(\gamma)), \text{ for } \gamma \in T(F).$$

In the real case, the Weyl denominator of Harish-Chandra's relative discrete series character formula is given by $\Delta^0(\gamma, \Delta^+)\rho(\tilde{\gamma})$, where $\gamma \in T(\mathbb{R})$ is regular and $\tilde{\gamma}$ is any lift of γ to the cover $T(\mathbb{R})_\rho$ of $T(\mathbb{R})$. Note that ρ is naturally a function on $T(\mathbb{R})_\rho$. Since $\rho(\tilde{\gamma})^2 = 2\rho(\gamma)$, $\rho(\tilde{\gamma})$ is canonically a square root of $2\rho(\gamma) = 2\rho(\Pi(\tilde{\gamma}))$, where $\Pi : T(\mathbb{R})_\rho \to T(\mathbb{R})$ is the canonical projection (see §3). That is, ρ is a canonical square root of $2\rho \circ \Pi$. Our Weyl denominator is a p-adic analogue of $\Delta^0(\gamma, \Delta^+)\rho(\tilde{\gamma})$, as we now explain.

If $\gamma \in T(F)$ is regular, note that $\Delta^0(\gamma, \Delta^+) \in E^*$. Let η be any unramified character of E^* whose restriction to F^* has order $[E : F]$ (these conditions on η will be used in the proofs of Propositions 9.3 and 9.5. For example, see (13) and (17).). We may now apply η to $\Delta^0(\gamma, \Delta^+)$. Similarly, for the p-adic version of $\rho(\tilde{\gamma})$, we wish to define a character η_ρ on $T(E)_\Gamma$ that will act as a "square root" of $\eta \circ (2\rho) \circ N$. In the case of $PGSp(4, F)$ we do this in an ad hoc fashion (see §9). Because of recent joint work of the first named author with David Roe (see [7]), we now have a conjectural way of defining η_ρ canonically, for an arbitrary connected reductive group, again assuming that E/F is unramified. Explicitly, in the language of [7], we may set $\eta_\rho = \boldsymbol{\mu}(T, \xi)$. If ξ is *minimal* (see [7]), we explicitly construct $\boldsymbol{\mu}(T, \xi)$ in an intrinsic way.

We can now define our p-adic analogue of Harish-Chandra's relative discrete series character formula. We first set $W_F(T) = N(G(E), T(F))/T(E)$ and $\mathscr{W}_F(T) = N(G(F), T(F))/T(F)$. where $N(A, B)$ denotes the normalizer of B in A. It is not difficult to see that both $W_F(T)$ and $\mathscr{W}_F(T)$ act on $T(E)_\Gamma$, and therefore on $\widehat{T(E)_\Gamma}$. Here $\widehat{T(E)_\Gamma}$ denotes the group of smooth complex-valued characters of $T(E)_\Gamma$. If $s \in W_F(T)$, we set $s_*\xi(\tilde{\gamma}) := \xi(s^{-1}\tilde{\gamma}s)$, for $\tilde{\gamma} \in T(E)_\Gamma$. Let $T(F)_{0,s}$ denote the set of strongly regular topologically semisimple elements of $T(F)$ (see [11, §7]), and let Z denote the center of G.

Recall our assumption that the Langlands parameter ϕ does not factor through a proper Levi subgroup of ${}^LG$ but does factor through the normalizer of a maximal torus in ${}^LG$. Moreover, we assume that some (hence every) maximal F-torus T of G in the stable conjugacy class attached to ϕ splits over some unramified extension E/F. As indicated above, the character ξ of $T(E)_\Gamma$ associated to ϕ depends on a choice of isomorphism implementing duality between T and $\hat{T}$. Varying this isomorphism changes ξ by an element of $W_F(T)$. (See §8 for more details.) We assume here that $\widehat{H}^0(\Gamma, T(E)) = 0$ for one (hence all) such T.

Conjecture 1.1. *(cf. [3, Definition 4.3])*

1. *Choose a character ξ of $T(E)_\Gamma$ associated to ϕ as above. There exists a function $\epsilon : \mathscr{W}_F(T) \to \{\pm 1\}$ and a constant $\epsilon(\xi, \Delta^+) \in \mathbb{C}^*$, depending on ξ and Δ^+, such that*

$$\Theta_\xi(\gamma) := \frac{\epsilon(\xi, \Delta^+)}{\eta(\Delta^0(\gamma, \Delta^+))\eta_\rho(\tilde{\gamma})} \sum_{n \in \mathscr{W}_F(T)} \epsilon(n) n_*\xi(\tilde{\gamma})$$

   agrees on $Z(F)T(F)_{0,s}$ with the character of a unique depth-zero supercuspidal representation $\pi(T, \xi)$ of $G(F)$ of the type considered in [11]. Here, $\tilde{\gamma}$ is any element of $T(E)_\Gamma$ such that $N(\tilde{\gamma}) = \gamma$ (the existence of $\tilde{\gamma}$ follows from our assumption that $\hat{H}^0(\Gamma, T(E)) = 0$).

2. *Set $L(\phi) := \{\pi(T, \xi)\}_{T,\xi}$, where T ranges over a set of representatives for the rational conjugacy classes of maximal F-tori in the stable class determined by ϕ, and for each such T, ξ ranges over the ($W_F(T)$-orbit of) characters of $T(E)_\Gamma$ arising from ϕ via the construction of Gross. Then the assignment $\phi \mapsto L(\phi)$ agrees with the local Langlands correspondence for $G(F)$.*

We note that this conjecture holds for $GL(\ell, F)$, where $p > 2\ell$ (see [6]). In §9, we will define a function η_ρ on $T(E)_\Gamma$ that is a canonical square root of $\eta \circ 2\rho \circ N$, in the case that $G(F) = PGSp(4, F)$. Our main theorem is the following.

Theorem 1.2. *This conjecture holds for the depth-zero supercuspidal L-packets of $PGSp(4, F)$ arising from the Debacker-Reeder construction. In this setting $\epsilon(n) = 1$ for all $n \in \mathscr{W}_F(T)$, and $\epsilon(\xi, \Delta^+) = 1$.*

The L-packets that arise in the above setting are of the following two forms. One family comprises singleton L-packets, the elements of which are supercuspidal representations associated (as in [11]) to elliptic maximal F-tori of $PGSp(4)$ having the quartic unramified extension of F as their splitting field. The other family consists of L-packets of size 2. The elements of each such packet are supercuspidal representations associated to stably, but not rationally, conjugate elliptic maximal F-tori which split over the quadratic unramified extension of F.

We would also like to note that in the case that $\hat{H}^0(\Gamma, T(E)) \neq 0$, the situation seems more difficult since our formula Θ_ξ is not defined on $Z(F)T(F)_{0,s}$ anymore, but namely on the image of $T(E)_\Gamma$ under the norm map. However, one might be able to remedy this with a prediction of central character, as in [13], for example. We would also like to note that there is a wide class of supercuspidal representations of p-adic groups that come from elliptic maximal tori that split over a ramified extension. In this situation, it is unclear how to define η and η_ρ, though this is the subject of future work.

We now present an outline of the paper. In §3, we provide the background material that we need from real groups as well as the motivation for our work. In §4, we review the theory of groups of type L due to Gross, which is an integral part of our local Langlands correspondence. In §5, we review the construction of [11]. In §6, we conduct a thorough analysis of the unramified elliptic tori of $PGSp(4, F)$. In §7, we explicitly determine the genuine characters of $T(E)_\Gamma$ arising from Gross's theory. In §8, we compare the construction of Gross to the construction of DeBacker/Reeder. In §9, we prove Theorem 1.2.

Remark 1.3. We believe that many of the results in this paper are generalizable in a way that does not involve large amounts of computations. As mentioned earlier, we have a conjectural way of generalizing η_ρ to an arbitrary unramified connected reductive group. Moreover, we believe that the results in §7 can be generalized without the need for computations. It remains for us to see whether the computations in §9 can be generalized.

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2 Notation

Now let F denote a nonarchimedean local field of characteristic zero. We let $\mathfrak{o}_F$ denote the ring of integers of F, $\mathfrak{p}_F$ its maximal ideal, $\mathfrak{f}$ the residue field of F, q the order of $\mathfrak{f}$, and p the characteristic of $\mathfrak{f}$. Let $\mathfrak{f}_\mathfrak{m}$ denote the degree m extension of $\mathfrak{f}$. We let ϖ denote a uniformizer of F. Let F^u denote the maximal unramified extension of F. Set $\Gamma_u = \mathrm{Gal}(F^u/F)$. We denote by $\mathcal{W}_F$ the Weil group of F, I_F the inertia subgroup of $\mathcal{W}_F$, I_F^+ the wild inertia subgroup of $\mathcal{W}_F$, and $\mathcal{W}_F^{\mathrm{ab}}$ the abelianization of $\mathcal{W}_F$. We denote by $\mathcal{W}'_F$ the Weil-Deligne group $\mathcal{W}_F \times \mathrm{SL}(2, \mathbb{C})$, we set $\mathcal{W}_t := \mathcal{W}_F/I_F^+$, and we set $I_t := I_F/I_F^+$. We fix an element $\Phi \in \mathrm{Gal}(\overline{F}/F)$ whose inverse induces the map $x \mapsto x^q$ on $\mathfrak{F} := \bar{\mathfrak{f}}$.

For any finite Galois extension E/F, let $N_{E/F} : E \to F$ denote the norm map. If E/F is a quadratic extension, we will sometimes denote the nontrivial Galois automorphism of E/F by $x \mapsto \bar{x}$. We let E_1 denote the quadratic unramified extension of F and we let E_2 denote the quartic unramified extension of F. Set $\Gamma_i = \mathrm{Gal}(E_i/F)$.

If A and B are abstract or algebraic groups, and if B is a subgroup of A, then we let $N(A, B)$ denote the normalizer of B in A. If B is a normal subgroup of A, we denote the image of $a \in A$ in A/B by $[a]$. For $a \in A$, $\mathrm{Int}(a)$ will denote the automorphism of A given by conjugation by a.

Suppose that G is a connected reductive group over an arbitrary field F, and $T \subset G$ a torus defined over F. We denote by $X^*(T)$ and $X_*(T)$ the character and cocharacter modules of T, respectively. If T happens to be unramified, and E/F is an unramified extension, then we define $T(\mathfrak{o}_E)$ to be the maximal bounded subgroup of $T(E)$. Let Δ^+ be a set of positive roots of G with respect to T. We set

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

For $\gamma \in T(F)$, define

$$\Delta^0(\gamma, \Delta^+) := \prod_{\alpha \in \Delta^+} (1 - \alpha^{-1}(\gamma)).$$

Denote by $W(G,T)$ the absolute Weyl group $N(G,T)/T$ of T in G, and set

$$\begin{aligned} W_F(G,T) &:= W(G,T)^{\mathrm{Gal}(\bar{F}/F)} \subset W(G,T) \\ \mathscr{W}_F(G,T) &:= N(G(F),T(F))/T(F) \subset W_F(G,T). \end{aligned}$$

The groups $W_F(G,T)$ and $\mathscr{W}_F(G,T)$ are the "large" and "small" Weyl groups as in [11, §2.11]. When the group G is understood from context, we will omit it from the notation, writing $W(T)$, $W_F(T)$, and $\mathscr{W}_F(T)$.

3 Background from real groups

In order to motivate the theory that we wish to develop for p-adic groups, we describe the corresponding theory over $\mathbb{R}$ upon which our work is based, that is, the the local Langlands correspondence for relative discrete series L-packets. More information can be found in [2].

3.1 Covers of Tori

Definition 3.1. Let G be a connected reductive group over $\mathbb{R}$, $T \subset G$ a torus over $\mathbb{R}$, and let Δ^+ be a set of positive roots of G with respect to T. Then $2\rho \in X^*(T)$. We define the *ρ-cover $T(\mathbb{R})_\rho$ of $T(\mathbb{R})$* as the fiber product (in the category of groups) of the homomorphisms $2\rho : T(\mathbb{R}) \to \mathbb{C}^*$ and the squaring map $\Upsilon : \mathbb{C}^* \to \mathbb{C}^*$ given by $\Upsilon(z) = z^2$. Thus, $T(\mathbb{R})_\rho = \{(t,\lambda) \in T(\mathbb{R}) \times \mathbb{C}^* : 2\rho(t) = \lambda^2\}$.

Although ρ is not necessarily a character of $T(\mathbb{R})$, it can naturally be viewed as a character of $T(\mathbb{R})_\rho$. Namely, in the commutative diagram

$$\begin{array}{ccc} T(\mathbb{R})_\rho & \xrightarrow{\Pi'} & \mathbb{C}^* \\ \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \Upsilon} \\ T(\mathbb{R}) & \xrightarrow{2\rho} & \mathbb{C}^* \end{array}$$

defining the fiber product, we have $\Pi'(\tilde{t})^2 = 2\rho(t)$, where $\Pi(\tilde{t}) = t$. Therefore, Π' is a character of $T(\mathbb{R})_\rho$ which is a canonical square root of $2\rho \circ \Pi$. Throughout this paper, we will write ρ instead of Π'.

The Weyl group $\mathscr{W}_\mathbb{R}(T) = N(G(\mathbb{R}), T(\mathbb{R}))/T(\mathbb{R})$ acts on $T(\mathbb{R})_\rho$ as follows: If $(t,\lambda) \in T(\mathbb{R})_\rho$ and $s \in \mathscr{W}_\mathbb{R}(T)$, then define

$$s(t,\lambda) := (st, (s^{-1}\rho - \rho)(t)\lambda) \tag{1}$$

Definition 3.2. A character $\tilde{\chi} : T(\mathbb{R})_\rho \to \mathbb{C}^*$ is *genuine* if it does not factor through Π.

Definition 3.3. A genuine character $\tilde{\chi}$ of $T(\mathbb{R})_\rho$ is called *regular* if ${}^s\tilde{\chi} \neq \tilde{\chi}$ for all nontrivial $s \in \mathscr{W}_{\mathbb{R}}(T)$ where ${}^s\tilde{\chi}(t,\lambda) := \tilde{\chi}(s^{-1}(t,\lambda))$.

3.2 Relative discrete series Langlands parameters and character formulas for real groups

In this section we will briefly describe the local Langlands correspondence for relative discrete series L-packets of real groups. Let G be a connected reductive group over $\mathbb{R}$ that contains a relatively compact maximal torus It is known that this is equivalent to $G(\mathbb{R})$ having relative discrete series representations.

Definition 3.4. Let t be an indeterminate and let k denote the rank of G. For $h \in G(\mathbb{R})$, define the Weyl denominator $D_G(h)$ by

$$\det(t + 1 - \mathrm{Ad}(h)) = D_G(h)t^k + (\text{terms of higher degree})$$

Then if Δ is the set of roots of T in G,

$$D_G(h) = \prod_{\alpha\in\Delta}(1 - \alpha(h)).$$

For a maximal torus $T \subset G$ defined over $\mathbb{R}$, define $\Delta^0(h, \Delta^+)$ as in §2, where Δ^+ is a set of positive roots of G with respect to T

We now present the classification of relative discrete series representations of $G(\mathbb{R})$ (see [3, Definition 4.3]).

Theorem 3.5. *(Harish-Chandra) Let G be a connected reductive group, defined over $\mathbb{R}$. Suppose that G contains a real Cartan subgroup T that is relatively compact. Let $\tilde{\chi}$ be a genuine character of $T(\mathbb{R})_\rho$ that is regular. Let $\epsilon(s) := (-1)^{\ell(s)}$ where $\ell(s)$ is the length of the Weyl group element $s \in \mathscr{W}_{\mathbb{R}}(T)$. Let $T(\mathbb{R})^{\mathrm{reg}}$ denote the regular set of $T(\mathbb{R})$. Then there exists a unique constant $\epsilon(\tilde{\chi}, \Delta^+) = \pm 1$, depending only on $\tilde{\chi}$ and Δ^+, and a unique relative discrete series representation of $G(\mathbb{R})$, denoted $\pi(\tilde{\chi})$, such that*

$$\Theta_{\pi(\tilde{\chi})}(h) = \frac{\epsilon(\tilde{\chi}, \Delta^+)}{\Delta^0(h, \Delta^+)\rho(\tilde{h})} \sum_{s\in\mathscr{W}_{\mathbb{R}}(T)} \epsilon(s)\tilde{\chi}({}^s\tilde{h}), \quad \text{for } h \in T(\mathbb{R})^{\mathrm{reg}}$$

where $\tilde{h} \in T(\mathbb{R})_\rho$ is any element such that $\Pi(\tilde{h}) = h$. Moreover, every relative discrete series character of $G(\mathbb{R})$ is of this form.

It is important to note that while the numerator and denominator of the character formula are functions on $T(\mathbb{R})_\rho$, the quotient factors to a function on $T(\mathbb{R})^{\mathrm{reg}}$.

We conclude the section by describing the local Langlands correspondence for relative discrete series representations of $G(\mathbb{R})$, assuming that $G(\mathbb{R})$ has relative discrete series. Fix a positive set of roots Δ^+ of G with respect to T. Let $\mathcal{W}_{\mathbb{R}}$ be the *Weil group* of $\mathbb{R}$. Let $\phi : \mathcal{W}_{\mathbb{R}} \to {}^LG$ be a relative discrete series Langlands parameter. The theory in [2] canonically attaches a genuine character $\tilde{\chi}$ of $T(\mathbb{R})_\rho$ to ϕ. Then the local Langlands correspondence for relative discrete series representations of $G(\mathbb{R})$ is given by attaching to $\tilde{\chi}$ the absolute Weyl group orbit of $\Theta_{\pi(\tilde{\chi})}$, where the absolute Weyl group acts on $\Theta_{\pi(\tilde{\chi})}$ via its action on $\tilde{\chi}$. The rest of the paper will be devoted to proving a p-adic analogue of this correspondence, for depth-zero supercuspidal L-packets of $PGSp(4, F)$.

4 Groups of type L

We now review the theory of "groups of type L" due to Benedict Gross. Let F be a field, F^{s} a separable closure, and T a torus defined over F that splits over an extension $E \subset F^s$. Let $\Gamma = \mathrm{Gal}(E/F)$. Let $X^*(T)$ be the character module of T and $X_*(T)$ the cocharacter module of T. Define $\hat{T} = X^*(T) \otimes \mathbb{C}^*$. The group Γ acts on $\hat{T}$ via its action on $X^*(T)$.

Definition 4.1. A *group of type* L is a group extension of Γ by $\hat{T}$.

Let D be such a group. Then we have an exact sequence

$$1 \to \hat{T} \to D \to \Gamma \to 1$$

We now describe how, to a given Langlands parameter

$$\phi : \mathcal{W}_F \to D,$$

we can naturally attach a character of $T(E)_\Gamma := T(E)/I_\Gamma(T(E))$, where $I_\Gamma(T(E)) = \{(1-\gamma)t \ : t \in T(E), \gamma \in \Gamma\}$. Restricting ϕ to $\mathcal{W}_E$ we get a homomorphism

$$\phi|_{\mathcal{W}_E} : \mathcal{W}_E \to \hat{T}$$

By the Langlands correspondence for tori, this gives us a character $\xi : T(E) \to \mathbb{C}^*$. Since $\phi|_{\mathcal{W}_E}$ extends to ϕ, one can see that

$$\xi(t^\sigma) = \xi(t) \text{ for all } \sigma \in \Gamma.$$

Therefore, $\xi(t^{\sigma-1}) = 1$ for all $\sigma \in \Gamma$. Thus, ξ is trivial on the augmentation ideal $I_\Gamma(T(E))$ and therefore ξ factors to a character

$$\xi : T(E)_\Gamma \to \mathbb{C}^*$$

Invariants and coinvariants are related by the norm map

$$N : T(E) \to T(F), \qquad t \mapsto \prod_{\xi \in \Gamma} \xi(t)$$

in the exact Tate cohomology sequence

$$1 \to \hat{H}^{-1}(\Gamma, T(E)) \to T(E)_\Gamma \xrightarrow{N} T(F) = T(E)^\Gamma \to \hat{H}^0(\Gamma, T(E)) \to 1, \tag{2}$$

noting that the norm map N factors to $T(E)_\Gamma$. We have thus constructed a character ξ of $T(E)_\Gamma$ from a Langlands parameter ϕ. We note that $T(E)_\Gamma$ is a cover of $N(T(E)_\Gamma)$, which is a subgroup of $T(F)$. It is sometimes the case that N is surjective, in which case ξ is then a character of $T(E)_\Gamma$, which is a cover of $T(F)$.

We will need the following result on Weyl group actions on $T(E)_\Gamma$ in §9.1.

Lemma 4.2. *Let G be a connected reductive F-group and let T be a maximal F-torus of G. Let E be the splitting field of T. Then the standard actions of $W_F(G,T)$ and $\mathscr{W}_F(G,T)$ on $T(E)$ factor to actions on $T(E)_\Gamma$.*

Proof. Since $\mathscr{W}_F(G,T) \subseteq W_F(G,T)$, it suffices to to prove the statement for the group $W_F(G,T)$. By [11, Lemma 2.11.2], $W_F(G,T) = N(G(E),T(F))/T(E)$. Let $n \in N(G(E),T(F))$. Then since n normalizes $T(F)$, n also normalizes $T(E)$. Therefore, we have that $N(G(E),T(F))/T(E)$ acts on $T(E)$.

We now have to show that $N(G(E),T(F))/T(E)$ sends the augmentation ideal

$$\{t\sigma(t)^{-1} : t \in T(E), \sigma \in \Gamma\}$$

to itself. Let $t \in T(E), \sigma \in \Gamma$ and let $w \in (N(G,T)/T)(F)$. Then

$$w\left(\frac{t}{\sigma(t)}\right) = \frac{w(t)}{w\sigma(t)} = \frac{w(t)}{\sigma(\sigma^{-1}(w(\sigma(t))))} = \frac{w(t)}{\sigma({}^{\sigma^{-1}}w)(t)} = \frac{w(t)}{\sigma(w(t))},$$

the last equality coming from the fact that $w \in (N(G,T)/T)(F)$, so w is fixed by Γ. □

5 Review of part of the construction of DeBacker and Reeder

In this section, we recall the part of the construction of DeBacker and Reeder that associates a character of an elliptic torus to a TRSELP.

Suppose G is an unramified connected reductive group defined over F. We fix $S \subset G$, an F^u-split maximal F-torus which is contained in a Borel F-subgroup. We let $X := X_*(S)$ be the character module of S, and let $W_o = W(G,S)$. We denote by ϑ the automorphism of X and $X \rtimes W_o$ induced by Φ, where $X \rtimes W_o$ is the extended affine Weyl group.

We now review some of the basic theory from [11]. Let $\hat{G}$ denote the complex dual group of G. Let $\hat{S} \subset \hat{G}$ be a maximal torus and fix an identification $\delta : X \xrightarrow{\sim} X^*(\hat{S})$. Fix a pinning $(\hat{S}, \hat{B}, \{x_{\alpha^*}\}_{\alpha^* \in \Phi^*})$ for $\hat{G}$ once and for all. The operator $\hat{\vartheta}$ on $X_*(\hat{S})$ dual to ϑ induces an automorphism of $\hat{S}$. There is a unique extension of $\hat{\vartheta}$ to an automorphism of $\hat{G}$, satisfying $\hat{\vartheta}(x_{\alpha_*}) = x_{\vartheta\cdot\alpha_*}$ (see [11, §3.2]). Following [11], we may form the semidirect product ${}^LG := \langle\hat{\vartheta}\rangle \ltimes \hat{G}$.

Definition 5.1. A Langlands parameter $\phi : \mathcal{W}'_F \to {}^LG$ is called a *tame regular semisimple elliptic Langlands parameter* (abbreviated TRSELP) if

1. ϕ is trivial on I_F^+,
2. The centralizer of $\phi(I_F)$ in $\hat{G}$ is a torus.
3. $C_{\hat{G}}(\phi)^o = (\hat{Z}^{\hat{\vartheta}})^o$, where $\hat{Z}$ denotes the center of $\hat{G}$, and where $C_{\hat{G}}(\phi)$ denotes the centralizer of ϕ in $\hat{G}$.

Condition (2) forces ϕ to be trivial on $SL(2,\mathbb{C}) \subset \mathcal{W}'_F$. Let $\hat{N} = N(\hat{G},\hat{S})$. After conjugating by an appropriate element of $\hat{G}$, we may assume that $\phi(I_F) \subset \hat{S}$ and $\phi(\Phi) = \hat{\vartheta}f$, for some $f \in \hat{N}$. Let $\hat{w}$ be the image of f in $\hat{W}_o = \hat{N}/\hat{S}$, and let w be the corresponding element of W_o under the natural identification of W_o with $\hat{W}_o$.

Let ϕ be a TRSELP with associated w and let σ be the automorphism $w\vartheta$ of S. Let $\hat{\sigma}$ be the automorphism of $\hat{S}$ dual to σ, and let n be the order of σ. We set $\hat{G}_{\mathrm{ab}} := \hat{G}/\hat{G}'$, where $\hat{G}'$ denotes the derived group of $\hat{G}$. Let ${}^LS_\sigma := \langle\hat{\sigma}\rangle \ltimes \hat{S}$. DeBacker and Reeder (see [11, §4]) associate to ϕ a $\hat{S}$-conjugacy class of Langlands parameters

$$\phi_S : \mathcal{W}_t \to {}^LS_\sigma \tag{3}$$

as follows. Set $\phi_S := \phi$ on I_F, and $\phi_S(\Phi) := \hat{\sigma} \ltimes \tau$ where $\tau \in \hat{S}$ is any element whose class in $\hat{S}/(1-\hat{\sigma})\hat{S}$ corresponds to the image of f in $\hat{G}_{\rm ab}/(1-\hat{\vartheta})\hat{G}_{\rm ab}$ under the bijection

$$\hat{S}/(1-\hat{\sigma})\hat{S} \xrightarrow{\sim} \hat{G}_{\rm ab}/(1-\hat{\vartheta})\hat{G}_{\rm ab} \tag{4}$$

In [11, §4], DeBacker and Reeder construct a canonical bijection between $\hat{S}$-conjugacy classes of admissible homomorphisms $\phi : \mathcal{W}_t \to {}^L S_\sigma$ and depth-zero characters of $S(F^u)^{\Phi_\sigma}$, where we identify $S(F^u) = X \otimes F^u$ and Φ_σ is the automorphism $\sigma \otimes \Phi^{-1}$. We briefly summarize this construction. Let $\mathbb{S} := X \otimes \mathfrak{F}^*$ and note that Φ_σ also acts on $\mathbb{S}$. Given automorphisms α, β of abelian groups A, B, respectively, let $\mathrm{Hom}_{\alpha,\beta}(A, B)$ denote the set of homomorphisms $f : A \to B$ such that $f \circ \alpha = \beta \circ f$. Composition with the norm map $N_\sigma(t) = t\Phi_\sigma(t)\Phi_\sigma^2(t) \cdots \Phi_\sigma^{n-1}(t)$ induces isomorphisms

$$\mathrm{Hom}(\mathbb{S}^{\Phi_\sigma}, \mathbb{C}^*) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(\mathbb{S}^{\Phi_\sigma^n}, \mathbb{C}^*) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_n^*, \mathbb{C}^*).$$

Moreover, the map $s \mapsto \chi_s$ gives an isomorphism

$$\mathrm{Hom}_{\Phi,\hat{\sigma}}(\mathfrak{f}_n^*, \hat{S}) \xrightarrow{\sim} \mathrm{Hom}_{\Phi_\sigma,\mathrm{Id}}(X \otimes \mathfrak{f}_n^*, \mathbb{C}^*),$$

where $\chi_s(\lambda \otimes a) := \lambda(s(a))$. The canonical projection $I_t \to \mathfrak{f}_m^*$ induces an isomorphism as Φ-modules $I_t/(1-\mathrm{Ad}(\Phi)^m)I_t \xrightarrow{\sim} \mathfrak{f}_m^*$. Since $\hat{\sigma}$ has order n, we have $\mathrm{Hom}_{\Phi,\hat{\sigma}}(\mathfrak{f}_n^*, \hat{S}) \cong \mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}(I_t, \hat{S})$. Therefore, the map $s \mapsto \chi_s$ is a canonical bijection

$$\mathrm{Hom}_{\mathrm{Ad}(\Phi),\hat{\sigma}}(I_t, \hat{S}) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{S}^{\Phi_\sigma}, \mathbb{C}^*).$$

Moreover, if ${}^0S(F^u)$ denotes the maximal bounded subgroup of $S(F^u)$, we have an isomorphism

$${}^0S(F^u)^{\Phi_\sigma} \times X^\sigma \xrightarrow{\sim} S(F^u)^{\Phi_\sigma}, \qquad (\gamma, \lambda) \mapsto \gamma\lambda(\varpi).$$

Finally, note that $\hat{S}/(1-\hat{\sigma})\hat{S}$ can be identified with the character group of X^σ, via the map taking $\tau \in \hat{S}/(1-\hat{\sigma})\hat{S}$ to $\chi_\tau \in \mathrm{Hom}(X^\sigma, \mathbb{C}^*)$, where $\chi_\tau(\lambda) := \lambda(\tau)$. Therefore, we have a canonical bijection $\phi \mapsto \chi_\phi$ between $\hat{S}$-conjugacy classes of admissible homomorphisms $\phi : \mathcal{W}_t \to {}^L S_\sigma$ and depth-zero characters of $S(F^u)^{\Phi_\sigma}$ given by

$$\chi_\phi := \chi_s \otimes \chi_\tau, \tag{5}$$

where $s := \phi|_{I_t}$, $\phi(\Phi) = \hat{\sigma} \ltimes \tau$.

6 Maximal tori in $GSp(4)$ and $PGSp(4)$

In this section, we describe the unramified elliptic maximal F-tori in $GSp(4)$ and $PGSp(4)$ as these are the tori that are relevant to the construction of L-packets in [11]. The torus

$$\dot{S} = \{\mathrm{diag}(x_1, x_2, x_3, x_4) : x_1x_4 = x_2x_3\}$$

is a split maximal torus of $GSp(4)$. Its Weyl group has order 8 and can be identified with the group of permutations of the elements $\{x_1, x_2, x_3, x_4\}$ that fix the relation $x_1x_4 = x_2x_3$. We choose a system of positive roots

$$\alpha(x) = x_1/x_2, \quad \beta(x) = x_2/x_3, \quad (\alpha+\beta)(x) = x_1/x_3, \quad (2\alpha+\beta)(x) = x_1/x_4, \tag{6}$$

where $x = \mathrm{diag}(x_1, x_2, x_3, x_4)$. All calculations in this paper will be performed with respect to this choice of positive roots. However, our results do not depend on this choice (see remark 9.7). We note that if ρ denotes half the sum of the positive roots, then $2\rho(x) = x_1^3/x_3^2 x_4$, where again $x = \mathrm{diag}(x_1, x_2, x_3, x_4)$.

Let $A = E_1 \times E_1$, and from now on let σ denote the generator of $\mathrm{Gal}(E_1/F)$. By [16, §1], there is a stable conjugacy class of elliptic maximal F-tori $\dot{T}$ of $GSp(4)$ such that

$$\dot{T}(F) \cong \{a \in A : a\sigma(a) \in F^*\},$$

where σ acts componentwise, and F^* is embedded diagonally as the set of pairs $\{(z, z) : z \in F^*\}$. Thus

$$\dot{T}(F) \cong \{(x, y) \in E_1^* \times E_1^* : N_{E_1/F}(x) = N_{E_1/F}(y)\}.$$

It follows from [10, Thm. 3.4.1] that there exists an F-torus $\dot{T}_0$ (resp. $\dot{T}_1$) in this stable class such that the maximal compact subgroup of $\dot{T}_0(F)$ (resp. $\dot{T}_1(F)$) is contained in a hyperspecial (resp. non-hyperspecial) maximal compact subgroup of $GSp(4, F)$. For $i = 0, 1$, there exists $g_i \in GSp(4, E_1)$ that conjugates $\dot{T}_i$ to $\dot{S}$ and which satisfies

$$\sigma(g_i^{-1}\mathrm{diag}(a, b, c, d)g_i) = g_i^{-1}\mathrm{diag}(\bar{d}, \bar{c}, \bar{b}, \bar{a})g_i^{-1}$$

for $\mathrm{diag}(a, b, c, d) \in \dot{S}(E_1)$. Thus, for $i = 0, 1$, we can and will identify $\dot{T}_i(E_1)$ with the group $\{(a, b, c, d) \in (E_1^*)^4 : ad = bc\}$, where $\sigma \in \Gamma_1$ acts via the formula $\sigma(a, b, c, d) = (\bar{d}, \bar{c}, \bar{b}, \bar{a})$ (recall that $\Gamma_1 = \mathrm{Gal}(E_1/F)$). Then $\dot{T}_i(F) = \dot{T}_i(E_1)^{\Gamma_1} = \{(a, c, \bar{c}, \bar{a}) : a\bar{a} = c\bar{c}\}$, which can further be identified with $\{(x, y) \in E_1^* \times E_1^* : N_{E_1/F}(x) = N_{E_1/F}(y)\}$.

From now on, denote by τ a fixed generator of $\mathrm{Gal}(E_2/F)$. Again by [16, §1], there is a stable conjugacy class of elliptic maximal F-tori $\dot{T}$ of $GSp(4)$ such that $\dot{T}(F) \cong \{x \in E_2 : x\tau^2(x) \in F^*\}$. Let $\dot{T}_2$ be such a torus. Let $g_2 \in GSp(4, E_2)$ be an element that conjugates $\dot{T}_2$ to $\dot{S}$. Then g_2 can be chosen to satisfy

$$\tau(g_2^{-1}\mathrm{diag}(a, b, c, d)g_2) = g_2^{-1}\mathrm{diag}(\tau(c), \tau(a), \tau(d), \tau(b))g_2.$$

for all $(a, b, c, d) \in \dot{S}(E_2)$. Thus $\dot{T}_2(F) = \dot{T}_2(E_2)^{\mathrm{Gal}(E_2/F)}$ is E_2-conjugate to the group of matrices of the form $\mathrm{diag}(a, \tau(a), \tau^3(a), \tau^2(a))$, where $a \in E_2^*$ satisfies $a\tau^2(a) \in F^*$, i.e., $N_{E_2/E_1}(a) \in F^*$.

Thus we will identify $\dot{T}_2(E_2)$ with the group $\{(a, b, c, d) \in (E_2^*)^4 : ad = bc\}$, where $\tau \in \Gamma_2$ acts via the formula $\tau(a, b, c, d) = (\tau(c), \tau(a), \tau(d), \tau(b))$. Then

$$\begin{aligned} \dot{T}_2(F) &= \dot{T}_2(E_2)^{\Gamma_2} \\ &= \{(a, \tau(a), \tau^3(a), \tau^2(a)) : a \in E_2^*, a\tau^2(a) \in F^*\}, \end{aligned}$$

which can be identified with $N_{E_2/E_1}^{-1}(F^*) = \{a \in E_2^* : a\tau^2(a) \in F^*\}$.

We denote by S the image of $\dot{S}$ in $PGSp(4)$. Thus the roots described earlier in this section can be viewed as roots of S.

6.1 The tori T_0 and T_1

Let T_0 (resp. T_1) be the image of $\dot{T}_0$ (resp. $\dot{T}_1$) in $PGSp(4)$. Then T_0 (resp. $\dot{T}_1$) is an elliptic F-torus such that $\dot{T}_0(F)$ (resp. $\dot{T}_1(F)$) is contained in a hyperspecial (resp. non-hyperspecial) maximal compact subgroup of $GSp(4, F)$. It follows from [10, Thm. 3.4.1,Cor. 4.3.2] that T_0 and T_1 are not rationally conjugate, and the union of their rational conjugacy classes is a stable conjugacy class of maximal F-tori in $PGSp(4)$. Let T be either T_0 or T_1. Then we can identify $T(E_1)$ with

$$\{(a,b,c,d)E_1^* : a,b,c,d \in E_1^*, ad = bc\},$$

where E_1^* is embedded diagonally. Then σ acts on $T(E_1)$ via $\sigma((a,b,c,d)E_1^*) = (\bar{d},\bar{c},\bar{b},\bar{a})E_1^*$. In this section, we will compute the Tate cohomology groups of T, and then give a simple description of the Tate cohomology exact sequence for T (see §4). In order to compute Galois invariants and coinvariants of $T(E_1)$, it is useful to further identify $T(E_1)$ with $E_1^* \times E_1^*$ as follows.

Lemma 6.1. *For $i = 0,1$, identify $T_i(E_1)$ with $\{(a,b,c,d)E_1^* : a,b,c,d \in E_1^*, ad = bc\}$ as above. There is a Galois-equivariant isomorphism*

$$\psi_i : T(E_1) \to E_1^* \times E_1^*$$

$$(a,b,c,d)E_1^* \mapsto (a/b, b/c)$$

where the Galois action on $E_1^* \times E_1^*$ is given by $\sigma(w,z) \mapsto (1/\bar{w}, 1/\bar{z})$. Thus

$$T_i(F) = T_i(E_1)^{\Gamma_1} \cong \ker(N_{E_1/F}) \times \ker(N_{E_1/F}).$$

Proof. The map is easily seen to be well defined. To show injectivity, suppose $\psi_i((a,b,c,d)E_1^*) = (1,1)$. Thus, $a = b = c$. Since $ad = bc$, we have $a = b = c = d$, so $(a,b,c,d)E_1^*$ is trivial in $T(E_1)$. The inverse of ψ_i is given by $(w,z) \mapsto (w,1,1/z,1/wz)E_1^*$. To show that ψ_i is Galois equivariant, note that

$$\begin{aligned}
\psi_i(\sigma((a,b,c,d)E_1^*)) &= \psi_i((\bar{d},\bar{c},\bar{b},\bar{a})E_1^*) \\
&= (\bar{d}/\bar{c}, \bar{c}/\bar{b}) \\
&= (\bar{b}/\bar{a}, \bar{c}/\bar{b}) \\
&= \sigma(a/b, b/c) \\
&= \sigma(\psi_i((a,b,c,d)E_1^*))
\end{aligned}$$

since $ad = bc$. The final statement now follows easily. □

Again, let T be either T_0 or T_1. We now consider the norm map $N : T(E_1) \to T(E_1)^{\Gamma_1}$ given in §4.

Lemma 6.2. $\hat{H}^0(\Gamma_1, T(E_1)) = 0$.

Proof. Recall that $\hat{H}^0(\Gamma_1, T(E_1)) = T(E_1)^{\Gamma_1}/N(T(E_1))$, where N is the norm map $T(E_1) \to T(F)$. Identifying $T(E_1)$ with $E_1^* \times E_1^*$ as in Lemma 6.1, we have $N(w,z) = (w/\bar{w}, z/\bar{z})$. Thus, by Hilbert's Theorem 90, $N(T(E_1)) = \ker(N_{E_1/F}) \times \ker(N_{E_1/F}) = T(E_1)^{\Gamma_1}$. □

Lemma 6.3. $\hat{H}^{-1}(\Gamma_1, T(E_1)) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Proof. Recall that $\hat{H}^{-1}(\Gamma_1, T(E_1)) = \ker(N)/I_{\Gamma_1}(T(E_1))$. Let $(w,z) \in E_1^* \times E_1^*$, which we identify with $T(E_1)$ as above. Since $N(w,z) = (w/\bar{w}, z/\bar{z})$, it follows that $\ker(N) = F^* \times F^*$.

Let $1-\sigma : T(E_1) \longrightarrow T(E_1)$ be the map $x \mapsto x/\sigma(x)$. Then $I_{\Gamma_1}(T(E_1)) = \mathrm{im}(1-\sigma)$. For $(x,y) \in E_1^* \times E_1^* = T(E_1)$, we have $(1-\sigma)(x,y) = (N_{E_1/F}(x), N_{E_1/F}(y))$, so $\mathrm{im}(1-\sigma) = N_{E_1/F}(E_1^*) \times N_{E_1/F}(E_1^*)$. Thus

$$\ker(N)/I_{\Gamma_1}(T(E_1)) = (F^*/(N_{E_1/F}(E_1^*)) \times (F^*/N_{E_1/F}(E_1^*)) \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}).$$

□

The map N factors through the group $T(E_1)_{\Gamma_1}$ of coinvariants to give a map $N : T(E_1)_{\Gamma_1} \to T(E_1)^{\Gamma_1}$. By the proof of the preceding theorem, we have that $T(E_1)_{\Gamma_1}$ can be identified with

$$\frac{E_1^* \times E_1^*}{(1-\sigma)(E_1^* \times E_1^*)} = \left(E_1^*/N_{E_1/F}(E_1^*)\right) \times \left(E_1^*/N_{E_1/F}(E_1^*)\right).$$

Recalling that $T(E_1)^{\Gamma_1} = \ker(N_{E_1/F}) \times \ker(N_{E_1/F})$, we have that $N : T(E_1)_{\Gamma_1} \to T(E_1)^{\Gamma_1}$ is given by

$$(wN_{E_1/F}(E_1^*), wN_{E_1/F}(E_1^*)) \mapsto (w/\bar{w}, z/\bar{z}).$$

Therefore, the exact sequence of Tate cohomology groups in §4 reduces in our case to the standard exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to E_1^*/N_{E_1/F}(E_1^*) \times E_1^*/N_{E_1/F}(E_1^*) \xrightarrow{N} \ker(N_{E_1/F}) \times \ker(N_{E_1/F}) \to 1$$

6.2 The torus T_2

In this section, we will compute the Tate cohomology groups of T_2, and then give a simple description of the Tate cohomology exact sequence for T_2.

Let T_2 be the image of $\dot{T}_2$ in $PGSp(4)$. If follows from [10, Cor. 4.3.2] that the rational conjugacy class of T_2 in $PGSp(4)$ is a stable class. We can identify $T_2(E_2)$ with $\{(a,b,c,d)E_2^* : a,b,c,d \in E_2^*, ad = bc\}$, where E_2^* is embedded in diagonally. Then τ acts on $T_2(E_2)$ via $\tau((a,b,c,d)E_2^*) = (\tau(c),\tau(a),\tau(d),\tau(b))E_2^*$. In order to compute Galois invariants and coinvariants of $T_2(E_2)$, the following further identification is useful.

Lemma 6.4. *Identifying $T_2(E_2)$ with $\{(a,b,c,d)E_2^* : a,b,c,d \in E_2^*, ad = bc\}$ as above, there is a Galois-equivariant isomorphism*

$$\psi_2 : T_2(E_2) \xrightarrow{\sim} E_2^* \times E_2^*$$
$$(a,b,c,d)E_2^* \mapsto (a/b, a/c),$$

where the Galois action on $E_2^* \times E_2^*$ is given by $\tau(x,y) := (\tau(y)^{-1}, \tau(x))$.

Proof. The map is clearly well defined, injective, and surjective as in the proof of Lemma 6.1. Moreover, if $(a,b,c,d)E_2^* \in T_2(E_2)$ (so that $ad = bc$), then

$$\begin{aligned}
\psi_2(\tau((a,b,c,d)E_2^*)) &= \psi_2((\tau(c),\tau(a),\tau(d),\tau(b))E_2^*) \\
&= (\tau(c/a), \tau(c/d)) \\
&= (\tau(c/a), \tau(a/b)) \\
&= \tau(a/b, a/c) \\
&= \tau(\psi_2((a,b,c,d)E_2^*)).
\end{aligned}$$

□

Lemma 6.5. $\hat{H}^0(\Gamma_2, T_2(E_2)) = 0$

Proof. Recall that $\hat{H}^0(\Gamma_2, T_2(E_2)) = T_2(E_2)^{\Gamma_2}/N(T_2(E_2))$, where N is the norm map $T_2(E_2) \to T_2(F)$ and $\Gamma_2 = \mathrm{Gal}(E_2/F)$. We identify $T_2(E_2)$ with $E_2^* \times E_2^*$ as in Proposition 6.4. Suppose that $(x, y) \in T_2(E_2)^{\Gamma_2}$. Then one easily sees that $y = \tau(x)$ and $x \in \ker(N_{E_2/E_1})$. Since τ^2 generates $\mathrm{Gal}(E_2/E_1)$, it follows from Hilbert's Theorem 90 that $x = w/\tau^2(w)$ for some $w \in E_2^*$. Then

$$\begin{aligned} N(w,1) &= (w,1)\cdot\tau(w,1)\cdot\tau^2(w,1)\cdot\tau^3(w,1) \\ &= (w,1)\cdot(1,\tau(w))\cdot(\tau^2(w)^{-1},1)\cdot(1,\tau^3(w)^{-1}) \\ &= (w/\tau^2(w), \tau(w/\tau^2(w))) \\ &= (x,\tau(x)) \end{aligned}$$

Thus $(x, y) \in N(T_2(E_2))$, so $N(T_2(E_2)) = T_2(E_2)^{\Gamma_2}$. □

Lemma 6.6. $\hat{H}^{-1}(\Gamma_2, T_2(E_2)) \cong \mathbb{Z}/2\mathbb{Z}$

Proof. Recall that $\hat{H}^{-1}(\Gamma_2, T_2(E_2)) = \ker(N)/I_{\Gamma_2}(T_2(E_2))$. Let $(w, z) \in E_2^* \times E_2^*$, which we identify with $T_2(E_2)$ as in Lemma 6.4. Note that

$$\begin{aligned} N(w,z) &= (w,z)\cdot\tau(w,z)\cdot\tau^2(w,z)\cdot\tau^3(w,z) \\ &= (w,z)\cdot(\tau(z)^{-1},\tau(w))\cdot(\tau^2(w)^{-1},\tau^2(z)^{-1})\cdot(\tau^3(z),\tau^3(w)^{-1}) \\ &= (w\tau(z)^{-1}\tau^2(w)^{-1}\tau^3(z), z\tau(w)\tau^2(z)^{-1}\tau^3(w)^{-1}). \end{aligned}$$

Thus $N(w, z) = (x, \tau(x))$, where $x = w\tau(z)^{-1}\tau^2(w)^{-1}\tau^3(z)$. The condition $N(w, z) = (1, 1)$ is therefore equivalent to $x = 1$, or $w/\tau(z) = \tau^2(w)/\tau^3(z) = \tau^2(w/\tau(z))$. It follows that $w/\tau(z) \in E_1^*$, so that $\tau(z) = w\alpha$ for some $\alpha \in E_1^*$. Thus

$$\ker(N) = \{(w, \tau^{-1}(w)\beta : w \in E_2^*, \beta \in E_1^*\}.$$

Let $\vartheta : T_2(E_2) \longrightarrow T_2(E_2)$ be the map $x \mapsto x/\tau(x)$. Then $I_{\Gamma_2}(T_2(E_2)) = \mathrm{im}(\vartheta)$. For $(x, y) \in E_2^* \times E_2^* = T_2(E_2)$, we have

$$\vartheta(x,y) = (x,y)\tau(x,y)^{-1} = (x,y)(\tau(y),\tau(x)^{-1}) = (x\tau(y), y/\tau(x)).$$

Making a change of variables by setting $w = x\tau(y)$, we get that

$$y/\tau(x) = y\tau^2(y)/\tau(w) = \tau^{-1}(w)\cdot\frac{N_{E_2/E_1}(y)}{\tau(w)\tau^{-1}(w)} = \tau^{-1}(w)\cdot N_{E_2/E_1}(y/\tau(w)).$$

It follows that

$$I_{\Gamma_2}(T_2(E_2)) = \{(w, \tau^{-1}(w)\beta) : w \in E_2^*, \beta \in N_{E_2/E_1}(E_2^*)\}, \tag{7}$$

and hence that $\ker(N)/\mathrm{im}(\vartheta) \cong \mathbb{Z}/2\mathbb{Z}$. □

Lemma 6.7. *There are natural isomorphisms*

$$\begin{aligned} T_2(E_2)_{\Gamma_2} &\cong E_2^*/N_{E_2/E_1}(E_2^*) \\ T_2(E_2)^{\Gamma_2} &\cong \ker(N_{E_2/E_1}). \end{aligned}$$

Identifying these pairs of isomorphic groups, the norm map $N : T_2(E_2)_{\Gamma_2} \to T_2(E_2)^{\Gamma_2}$ is given by

$$wN_{E_2/E_1}(E_2^*) \mapsto w/\tau^2(w) \quad \textit{for } w \in E_2^*.$$

Proof. Identify $T_2(E_2)$ with $E_2^* \times E_2^*$ as above. Using (7), we see that the surjective homomorphism $T_2(E_2) \to E_2^*$ defined by $(w, z) \mapsto w\tau(z)^{-1}$ maps $I_{\Gamma_2}(T_2(E_2))$ onto $N_{E_2/E_1}(E_2^*)$. It therefore defines an isomorphism $T_2(E_2)_{\Gamma_2} \cong E_2^*/N_{E_2/E_1}(E_2^*)$. Moreover, by the proof of Lemma 6.5, we have

$$T_2(E_2)^{\Gamma_2} = \{(x, \tau(x)) : x \in \ker(N_{E_2/E_1})\} \cong \ker(N_{E_2/E_1})$$

via the map $(x, \tau(x)) \mapsto x$.

Let $(w, 1) \in E_2^* \times E_2^*$, which we identify with $T_2(E_2)$ as above. Let $\overline{(w, 1)}$ be the image of $(w, 1)$ in $T_2(E_2)_{\Gamma_2}$. As in the proof of Lemma 6.5,

$$N\overline{(w, 1)} = (w/\tau^2(w), \tau(w/\tau^2(w))).$$

Under the isomorphisms in the preceding paragraph, $\overline{(w, 1)} \in T_2(E_2)_{\Gamma_2}$ corresponds to $wN_{E_2/E_1}(E_2^*) \in E_2^*/N_{E_2/E_1}(E_2^*)$, while $(w/\tau^2(w), \tau(w/\tau^2(w))) \in T_2(E_2)^{\Gamma_2}$ corresponds to $w/\tau^2(w) \in \ker(N_{E_2/E_1})$. □

Therefore, we have shown that the exact sequence of Tate cohomology groups in §4 reduces therefore in our case to the standard exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to E_2^*/N_{E_2/E_1}(E_2^*) \xrightarrow{N} \ker(N_{E_2/E_1}) \to 1$$

$$wN_{E_2/E_1}(E_2^*) \mapsto w/\tau^2(w)$$

7 Characters of covers of tori associated to TRSELPs

From now on, unless otherwise specified G will denote the group $PGSp(4)$. Let $\phi : \mathcal{W}_F \to {}^LG$ be a TRSELP for $G(F) = PGSp(4, F)$. Note that ${}^LG = \mathrm{Spin}(5, \mathbb{C}) \times \langle\hat{\vartheta}\rangle$. Recall that by the theory of groups of type L, ϕ naturally gives rise to a character ξ of the group of coinvariants $T(E)_\Gamma$ of a elliptic maximal F-torus T that splits over some extension E of F, where $\Gamma = \mathrm{Gal}(E/F)$. In this section, we explicitly compute the restriction of ξ to $\hat{H}^{-1}(\Gamma, T(E))$ (see §4).

Given a root α_* of $\hat{S}$ in $\mathrm{Spin}(5, \mathbb{C})$, we denote by w_{α_*} the reflection in the Weyl group of $\hat{S}$ in $\mathrm{Spin}(5, \mathbb{C})$ through α_*. For each such α_*, there is an associated homomorphism $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{Spin}(5, \mathbb{C})$. Let n_{α_*} be the image of the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ under this map. Then n_{α_*} lies in the normalizer $N(\hat{G}, \hat{S})$ of $\hat{S}$ in $\mathrm{Spin}(5, \mathbb{C})$, and its image in the Weyl group of $\hat{S}$ is w_{α_*}. Let $\alpha_*^\vee$ denote the coroot of $\hat{S}$ corresponding to α_*. Then

$$n_{\alpha_*}^2 = \alpha_*^\vee(-1). \tag{8}$$

We have the following commutation relations for all roots α_*, β_* and $t \in \mathbb{C}$ (see [9, §7.2])

$$n_{\alpha_*}\beta_*^\vee(t)n_{\alpha_*}^{-1} = (w_{\alpha_*}(\beta_*))^\vee(t) \tag{9}$$

$$n_{\alpha_*}n_{\beta_*}n_{\alpha_*}^{-1} = (w_{\alpha_*}(\beta_*))^\vee(\eta_{\alpha_*,\beta_*}) \cdot n_{w_{\alpha_*}(\beta_*)} \quad \text{for a certain } \eta_{\alpha_*,\beta_*} \in \mathbb{C}. \tag{10}$$

We will henceforth denote by α_* and β_*, respectively, the long and short simple roots of $\hat{S}$ in $\mathrm{Spin}(5, \mathbb{C})$ (with respect to $\hat{B}$).

Lemma 7.1. *Let $\hat{n} = n_{\alpha_*}n_{\beta_*}n_{\alpha_*}n_{\beta_*}$. Then $\hat{n}^2 = \beta_*^\vee(-1)$.*

Proof. Observe that

$$\begin{aligned}\hat{n}^2 &= n_{\alpha_*}n_{\beta_*}n_{\alpha_*}n_{\beta_*}n_{\alpha_*}n_{\beta_*}n_{\alpha_*}n_{\beta_*}\\ &= n_{\alpha_*}n_{\beta_*}\cdot(n_{\alpha_*}n_{\beta_*}n_{\alpha_*}^{-1})\cdot n_{\alpha_*}^2\cdot n_{\beta_*}n_{\alpha_*}n_{\beta_*}\\ &= n_{\alpha_*}n_{\beta_*}\cdot(\alpha_*+\beta_*)^\vee(\eta_{\alpha_*,\beta_*})\cdot n_{\alpha_*+\beta_*}\cdot\alpha_*^\vee(-1)\cdot n_{\beta_*}n_{\alpha_*}n_{\beta_*}\end{aligned}$$

by (8) and (10), since $w_{\alpha_*}(\beta_*)=\alpha_*+\beta_*$. The above expression is equal to

$$\begin{aligned}&n_{\alpha_*}\cdot\Big(n_{\beta_*}\left((\alpha_*+\beta_*)^\vee(\eta_{\alpha_*,\beta_*})\cdot n_{\alpha_*+\beta_*}\cdot\alpha_*^\vee(-1)\right)n_{\beta_*}^{-1}\Big)\cdot n_{\beta_*}^2\cdot n_{\alpha_*}n_{\beta_*}\\ &\quad= n_{\alpha_*}\cdot\left((\alpha_*+\beta_*)^\vee(\eta_{\alpha_*,\beta_*})\cdot(\alpha_*+\beta_*)^\vee(\eta_{\beta_*,\alpha_*+\beta_*})\cdot n_{\alpha_*+\beta_*}\cdot(\alpha_*+2\beta_*)^\vee(-1)\right)\\ &\quad\quad\cdot\beta_*^\vee(-1)\cdot n_{\alpha_*}n_{\beta_*} \qquad (11)\end{aligned}$$

again by (8), (9), and (10), since $w_{\beta_*}(\alpha_*+\beta_*)=\alpha_*+\beta_*$ and $w_{\beta_*}(\alpha_*)=\alpha_*+2\beta_*$. But (11) can be rewritten as

$$\begin{aligned}&n_{\alpha_*}\left((\alpha_*+\beta_*)^\vee(\eta_{\alpha_*,\beta_*}\ \eta_{\beta_*,\alpha_*+\beta_*})\cdot n_{\alpha_*+\beta_*}\cdot(\alpha_*+2\beta_*)^\vee(-1)\cdot\beta_*^\vee(-1)\right)n_{\alpha_*}^{-1}\cdot n_{\alpha_*}^2\cdot n_{\beta_*}\\ &\quad= \beta_*^\vee(\eta_{\alpha_*,\beta_*}\ \eta_{\beta_*,\alpha_*+\beta_*})\cdot\beta_*^\vee(\eta_{\alpha_*,\alpha_*+\beta_*})\cdot n_{\beta_*}\cdot(\alpha_*+2\beta_*)^\vee(-1)\\ &\quad\quad\cdot(\alpha_*+\beta_*)^\vee(-1)\cdot\alpha_*^\vee(-1)\cdot n_{\beta_*}, \qquad (12)\end{aligned}$$

since

$$w_{\alpha_*}(\alpha_*+\beta_*)=\beta_*,\quad w_{\alpha_*}(\alpha_*+2\beta_*)=\alpha_*+2\beta_*,\quad w_{\alpha_*}(\beta_*)=\alpha_*+\beta_*.$$

We may re-express (12) as

$$\begin{aligned}&\beta_*^\vee(\eta_{\alpha_*,\beta_*}\ \eta_{\beta_*,\alpha_*+\beta_*}\ \eta_{\alpha_*,\alpha_*+\beta_*})\cdot n_{\beta_*}\left((\alpha_*+2\beta_*)^\vee(-1)\cdot(\alpha_*+\beta_*)^\vee(-1)\cdot\alpha_*^\vee(-1)\right)n_{\beta_*}^{-1}\cdot n_{\beta_*}^2\\ &\quad= \beta_*^\vee(\eta_{\alpha_*,\beta_*}\ \eta_{\beta_*,\alpha_*+\beta_*}\ \eta_{\alpha_*,\alpha_*+\beta_*})\cdot\alpha_*^\vee(-1)\cdot(\alpha_*+\beta_*)^\vee(-1)\cdot(\alpha_*+2\beta_*)^\vee(-1)\cdot\beta_*^\vee(-1),\end{aligned}$$

as above, noting that $w_{\beta_*}(\alpha_*+2\beta_*)=\alpha_*$. One checks easily that $(\alpha_*+\beta_*)^\vee=2\alpha_*^\vee+\beta_*^\vee$ and $(\alpha_*+2\beta_*)^\vee=\alpha_*^\vee+\beta_*^\vee$. Therefore, the preceding displayed expression equals

$$\begin{aligned}&\beta_*^\vee(-\eta_{\alpha_*,\beta_*}\ \eta_{\beta_*,\alpha_*+\beta_*}\ \eta_{\alpha_*,\alpha_*+\beta_*})\cdot\alpha_*^\vee(-1)\cdot\left[2\alpha_*^\vee(-1)\cdot\beta_*^\vee(-1)\right]\cdot\left[\alpha_*^\vee(-1)\beta_*^\vee(-1)\right]\\ &\quad= \beta_*^\vee(-\eta_{\alpha_*,\beta_*}\ \eta_{\beta_*,\alpha_*+\beta_*}\ \eta_{\alpha_*,\alpha_*+\beta_*})\end{aligned}$$

Basic results on Chevalley groups (see [9]) allow one to compute that

$$-\eta_{\alpha_*,\beta_*}\ \eta_{\beta_*,\alpha_*+\beta_*}\ \eta_{\alpha_*,\alpha_*+\beta_*}=-1,$$

concluding the proof. □

Recall that our parameter satisfies $\phi(I_F)\subset\hat{S}$ and $\phi(\Phi)=\hat{\vartheta}f$, for some $f\in N_{\hat{G}}(\hat{S})$. The Weyl group of $\mathrm{Spin}(5,\mathbb{C})$ is the dihedral group of order 8. If we set $r=w_{\alpha_*}w_{\beta_*}$, then for ϕ to be a TRSELP, it is necessary that $\hat{w}$ must be a nontrivial element of $\langle r\rangle$. Therefore, without loss of generality, we may take $\hat{w}$ to be r or r^2 (since r^3 is conjugate to r in the Weyl group of $\mathrm{Spin}(5,\mathbb{C})$, so replacing r by r^3 would yield an equivalent TRSELP).

Definition 7.2. We say that a TRSELP ϕ is *of type* r if $\hat{w}=r$. We say that a TRSELP ϕ is *of type* r^2 if $\hat{w}=r^2$.

7.1 Parameters of type r^2.

Let ϕ be a TRSELP of type r^2. We first note that this type of TRSELP is associated to the stable conjugacy class of the tori T_0 and T_1, in the sense that these are the representatives for the rational classes of elliptic tori on which the action of the Galois group is obtained by twisting the action on S by the Weyl group element r^2.

Let T be either T_0 or T_1, and correspondingly, let $g \in PGSp(4, E_1)$ be either the element g_0 or g_1 in $GSp(4, E_1)$ from §6 (whose image in $PGSp(4, E_1)$ conjugates T to the image S of $\dot{S}$ in $PGSp(4)$), and let ψ be the isomorphism ψ_0 or ψ_1 from Lemma 6.1. Via conjugation by g, we can transfer the roots α and β of S to roots of T (which we will also denote by α and β to ease notation). Note that

$$\psi((a,b,c,d)E_1^*) = (a/b, b/c) = (\alpha((a,b,c,d)E_1^*), \beta((a,b,c,d)E_1^*)).$$

There is a compatible identification of $\hat{S}$ with $\mathbb{C}^* \times \mathbb{C}^*$, as follows. First, recall that we have fixed a duality map $\delta : X_*(\hat{S}) \xrightarrow{\sim} X^*(S)$. Composing δ with conjugation by g gives us a duality map $\delta' : X^*(T) \xrightarrow{\sim} X_*(\hat{S})$. Under this map, the action of the generator σ of Γ_1 on $X^*(T)$ corresponds to the action of r^2 on $X_*(\hat{S})$ (via the scalar -1). We now use the images of the simple roots α and β of $PGSp(4)$ under δ' to decompose $\hat{S}$ as a direct product. We may assume that these images are the coroots $\alpha_*^\vee$ and $\beta_*^\vee$ described above (after possibly altering our fixed pinning, if necessary). Explicitly, the isomorphism $\mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\sim} \hat{S}$ is given by

$$(w, z) \mapsto \alpha_*^\vee(w)\beta_*^\vee(z).$$

Via the duality map δ', the action of Γ_1 on T transfers to one on $\hat{S} \cong \mathbb{C}^* \times \mathbb{C}^*$ given by $\sigma(w, z) = (1/w, 1/z)$.

By the theory in §4, ϕ naturally gives rise to a character ξ of $T(E_1)_{\Gamma_1}$. We wish to compute $\xi|_{\hat{H}^{-1}(\Gamma_1, T_1(E_1))}$ explicitly.

By Lemma 6.2 and (2), we have an exact sequence

$$1 \to \hat{H}^{-1}(\Gamma_1, T(E_1)) \to T(E_1)_{\Gamma_1} \to T(F) \to 1.$$

Since $\phi|_{\mathcal{W}_{E_1}}$ has image in $\hat{S}$ and the abelianization of $\mathcal{W}_{E_1}$ is naturally isomorphic to E_1^*, $\phi|_{\mathcal{W}_{E_1}}$ factors through a map $\phi' : E_1^* \to \hat{S}$. Via the above identification of $\hat{S}$ with $\mathbb{C}^* \times \mathbb{C}^*$, we get a canonical map

$$E_1^* \to \mathbb{C}^* \times \mathbb{C}^*$$

$$x \mapsto (\xi_1(x), \xi_2(x))$$

The local Langlands correspondence for tori says that $\xi(w, z) = \xi_1(w)\xi_2(z)$ for all $w, z \in E_1^*$, where we view ξ as a character of $E_1^* \times E_1^*$ via the identification of $T(E_1)$ with $E_1^* \times E_1^*$ in §6.1. Under this identification, $\hat{H}^{-1}(\Gamma_1, T(E_1)) = F^*/N_{E_1/F}(E_1^*) \times F^*/N_{E_1/F}(E_1^*)$. Therefore, we wish to compute $\xi(\varpi, \varpi), \xi(\varpi, 1)$, and $\xi(1, \varpi)$.

Proposition 7.3. *We have $\xi(\varpi, \varpi) = -1$, $\xi(\varpi, 1) = 1$, and $\xi(1, \varpi) = -1$. Therefore, ξ factors to a genuine character of the two-fold cover $\widetilde{T(F)} := T(E_1)_{\Gamma_1}/\langle(\varpi, 1)\rangle$ of $T(F)$.*

Proof. Recall that the Artin map $\mathcal{W}_{E_1} \to E_1^*$ sends Φ^2 to ϖ. Write $\phi(\Phi^2) = \phi'(\varpi)$ as $\alpha_*^\vee(a)\beta_*^\vee(b)$, for some $a, b \in \mathbb{C}^*$. The above discussion allows us to conclude that $\xi(\varpi, \varpi) = ab$, $\xi(\varpi, 1) = a$, and $\xi(1, \varpi) = b$. We must therefore calculate $\phi(\Phi^2)$.

Since the element r^2 of the Weyl group of $\hat{S}$ can be written $w_{\alpha_*} w_{\beta_*} w_{\alpha_*} w_{\beta_*}$, the element $\hat{n} = n_{\alpha_*} n_{\beta_*} n_{\alpha_*} n_{\beta_*}$ lies in the preimage of r^2 in $N(\hat{G}, \hat{S})$.

We first assume that $f = \hat{n}$, and then we will show that our result is independent of choice of lift of r^2. By Lemma 7.1, we have $f^2 = \beta_*^\vee(-1) = \alpha_*^\vee(1)\beta_*^\vee(-1)$. Therefore, the claim follows in the case that $f = \hat{n}$.

Now suppose $f = \hat{t}\hat{n}$, for some $\hat{t} \in \hat{S}$. Then

$$\phi(\Phi^2) = \hat{t}\hat{n}\hat{t}\hat{n} = \hat{t}\hat{n}\hat{t}\hat{n}^{-1}\hat{n}^2.$$

We need to write $\hat{t}\hat{n}\hat{t}\hat{n}^{-1}\hat{n}^2$ in the form $\alpha_*^\vee(a)\beta_*^\vee(b)$ for some $a, b \in \mathbb{C}^*$. Note that $\hat{n}\hat{t}\hat{n}^{-1} = w_{\alpha_*} w_{\beta_*} w_{\alpha_*} w_{\beta_*}(\hat{t})$. Since $\hat{w}$ acts via $(w, z) \mapsto (1/w, 1/z)$ on $\hat{S}$, it follows that $\hat{t}\hat{n}\hat{t}\hat{n}^{-1} = 1$. In particular, $\phi(\Phi^2) = \alpha_*^\vee(1)\beta_*^\vee(-1)$, so we have our result. □

7.2 Parameters of type r.

Let ϕ be a TRSELP of type r. Note that this type of TRSELP is associated to the stable conjugacy class of the torus T_2, in the sense that this is an elliptic torus on which the action of the Galois group is obtained by twisting the action on S by the Weyl group element r.

The element r acts on the module $X_*(\hat{S}) \cong \mathbb{Z} \times \mathbb{Z}$ by $(a, b) \mapsto (-b, a)$. As in §7.1, we transfer the roots α and β of S to roots of T_2 via conjugation by the element g_2 of §6. Note that we have identified $T_2(E_2)$ with $E_2^* \times E_2^*$ in §6.2 via the roots α and $\alpha + \beta$ of T_2. More precisely, our identification is

$$\psi_2 : T_2(E_2) \xrightarrow{\sim} E_2^* \times E_2^*$$
$$(a, b, c, d)E_2^* \mapsto (a/b, a/c) = (\alpha((a, b, c, d)E_2^*), (\alpha + \beta)((a, b, c, d)E_2^*)).$$

As in §7.1, there is a compatible identification of $\hat{S}$ with $\mathbb{C}^* \times \mathbb{C}^*$, as follows. We first compose the duality map δ and conjugation by the element g_2 from §6 to get a duality map $\delta' : X^*(T_2) \xrightarrow{\sim} X_*(\hat{S})$ as in §7.1. We then use the images of α and $\alpha + \beta$ of $PGSp(4)$ under δ' to decompose $\hat{S}$ as a direct product. Recall from §7.1 that $\delta'(\alpha) = \alpha_*^\vee$ and $\delta'(\beta) = \beta_*^\vee$. Then explicitly, the map $\mathbb{C}^* \times \mathbb{C}^* \xrightarrow{\sim} \hat{S}$ is given by

$$(w, z) \mapsto \alpha_*^\vee(w)(\alpha_*^\vee + \beta_*^\vee)(z) = \alpha_*^\vee(wz)\beta_*^\vee(z).$$

Via the duality map δ', the action of Γ_2 on T_2 transfers to one on $\hat{S} \cong \mathbb{C}^* \times \mathbb{C}^*$ given by $\tau(w, z) = (1/z, w)$. As in §7.1, we wish to compute the restriction to $\hat{H}^{-1}(\Gamma_2, T_2(E_2))$ of the character ξ attached to ϕ by the theory in §4.

By Lemma 6.5 and (2), Recall the exact sequence

$$1 \to \hat{H}^{-1}(\Gamma_2, T_2(E_2)) \to T_2(E_2)_{\Gamma_2} \to T_2(F) \to 1.$$

Since $\phi|_{\mathcal{W}_{E_2}}$ has image in $\hat{S}$, it follows that $\phi|_{\mathcal{W}_{E_2}}$ factors through a map $\phi' : E_2^* \to \hat{S}$, as in §7.1. Via the above identification of $\hat{S}$ with $\mathbb{C}^* \times \mathbb{C}^*$, we get a canonical map

$$E_2^* \to \mathbb{C}^* \times \mathbb{C}^*$$
$$x \mapsto (\xi_1(x), \xi_2(x))$$

The local Langlands correspondence for tori says that $\xi(w, z) = \xi_1(w)\xi_2(z)$ for all $w, z \in E_2^*$, where we view ξ as a character of $E_2^* \times E_2^*$ via the identification of $T_2(E_2)$ with $E_2^* \times E_2^*$ in §6.2. Under this identification of $T_2(E_2)$ with $E_2^* \times E_2^*$, the element $(\varpi, 1)$ represents the nontrivial class in $\hat{H}^{-1}(\Gamma_2, T_2(E_2))$ (see lemma 6.6). Thus, we wish to compute $\xi(\varpi, 1)$.

Proposition 7.4. *We have $\xi(\varpi, 1) = -1$. Therefore, ξ is a genuine character of the two-fold cover $\widetilde{T_2(F)} = T_2(E_2)_{\Gamma_2}$ of $T_2(F)$.*

Proof. Recall that the Artin map $\mathcal{W}_{E_2} \to E_2^*$ sends Φ^4 to ϖ. Write $\phi(\Phi^4) = \phi'(\varpi)$ as $\alpha_*^\vee(a)(\alpha_*^\vee + \beta_*^\vee)(b)$, for some $a, b \in \mathbb{C}^*$. The above discussion allows us to conclude that $\xi(\varpi, 1) = a$. We must therefore calculate $\phi(\Phi^4)$.

The element $\hat{m} = n_{\alpha_*} n_{\beta_*}$ lies in the preimage of r in $N(\hat{G}, \hat{S})$. We first assume that $f = \hat{m}$, and then we will show that our result is independent of choice of lift of r.

Note that $\hat{m}^4 = n_{\alpha_*} n_{\beta_*} n_{\alpha_*} n_{\beta_*} n_{\alpha_*} n_{\beta_*} n_{\alpha_*} n_{\beta_*}$. This is precisely the expression that was calculated in the proof of Lemma 7.1 to equal $\beta^\vee(-1)$. Therefore, in the case that $f = \hat{m}$, we have our result, since $\beta_*^\vee(-1) = \alpha_*^\vee(-1)(\alpha_*^\vee + \beta_*^\vee)(-1)$, so that $\xi(\varpi, 1) = -1$.

Now suppose $f = \hat{t}\hat{m}$, for some $\hat{t} \in \hat{S}$. Then

$$\phi(\Phi^4) = (\hat{t}\hat{m})^4 = \hat{t}\hat{m}\hat{t}\hat{m}^{-1}\hat{m}^2\hat{t}\hat{m}^{-2}\hat{m}^3\hat{t}\hat{m}^{-3}\hat{m}^4.$$

Note that $\hat{w}$ acts by $(a, b) \mapsto (-b, a)$ on $X_*(T)$. It therefore must act via $(w, z) \mapsto (1/z, w)$ on $\hat{S}$. It follows that if $\hat{t} = (w, z)$, then $\hat{t}\hat{m}\hat{t}\hat{m}^{-1}\hat{m}^2\hat{t}\hat{m}^{-2}\hat{m}^3\hat{t}\hat{m}^{-3} = (w, z)(1/z, w)(1/w, 1/z)(z, 1/w) = 1$. Therefore, $\phi(\Phi^4) = \hat{m}^4 = \alpha_*^\vee(-1)(\alpha_*^\vee + \beta_*^\vee)(-1)$, so we finally have our result. □

Remark 7.5. To summarize, we have calculated in each case the restriction of ξ to $\hat{H}^{-1}(\Gamma_i, T_i(E_i))$ by computing powers of $\phi(\Phi)$. On the other hand, one can see that the computation of these powers of $\phi(\Phi)$ are also precisely the data needed to compute the cohomology classes of the groups of type L that arise for each Langlands parameter. In fact, Gross has predicted a canonical link between $\xi|_{\hat{H}^{-1}}$ and the cohomology class of the group of type L as follows. Recalling the notation of §4, there is a canonical sequence of isomorphisms

$$H^2(\Gamma, \hat{T}) \cong H^3(\Gamma, X^*(T)) \cong \hat{H}^{-3}(\Gamma, X_*(T))^* \cong \hat{H}^{-1}(\Gamma, T(E))^*,$$

where $\hat{H}^{-1}(\Gamma, T(E))^*$ denotes the dual of $\hat{H}^{-1}(\Gamma, T(E))$. These isomorphisms are given by the exponential sequence

$$1 \to X^*(T) \to X^*(T) \otimes \mathbb{C} \to \hat{T} \to 1,$$

the universal coefficients theorem, and Tate duality, respectively. Therefore, if D is a group of type L, the composition of the above isomorphisms canonically gives a homomorphism

$$c_D : \hat{H}^{-1}(\Gamma, T(E)) \to \mathbb{C}^*.$$

Gross has predicted that if $\phi : \mathcal{W}_F \to D$ is a Langlands parameter, and if $\xi \in \widehat{T(E)_\Gamma}$ is the character that ϕ gives rise to as in §4, then the restriction of ξ to $\hat{H}^{-1}(\Gamma, T(E))$ is equal to c_D.

8 Relating the constructions of Gross and DeBacker–Reeder

We consider here a TRSELP ϕ for any unramified connected reductive group G. Let w be the Weyl group element associated to ϕ, and set $\sigma = w\vartheta$ as in §5. Let S, Φ_σ be as in §5. Let S_σ denote the torus S with Galois action given by Φ_σ. Let E be the splitting field of S_σ, and set $\Gamma = \mathrm{Gal}(E/F)$. Then ϕ has image inside a group of type L for S_σ. Let χ_ϕ be the character of $S_\sigma(F)$ that DeBacker and Reeder attach to ϕ (see §5).

We have the exact sequence

$$1 \to \hat{H}^{-1}(\Gamma, S_\sigma(E)) \to S_\sigma(E)_\Gamma \to S_\sigma(F) \to \hat{H}^0(\Gamma, S_\sigma(E)) \to 1.$$

Let Ω_ϕ be the character of $S_\sigma(E)_\Gamma$ associated to ϕ via the theory of groups of type L (see §4). Note that the above exact sequence restricts to an exact sequence

$$1 \to \hat{H}^{-1}(\Gamma, S_\sigma(\mathfrak{o}_E)) \to S_\sigma(\mathfrak{o}_E)_\Gamma \to S_\sigma(\mathfrak{o}_F) \to \hat{H}^0(\Gamma, S_\sigma(\mathfrak{o}_E)) \to 1$$

Moreover, one can show using a profinite version of Lang's theorem and various results about tori over finite fields, that since S_σ is unramified, $\hat{H}^{-1}(\Gamma, S_\sigma(\mathfrak{o}_E)) = \hat{H}^0(\Gamma, S_\sigma(\mathfrak{o}_E)) = 1$ (see [7]). Therefore, the map

$$S_\sigma(\mathfrak{o}_E)_\Gamma \xrightarrow{N} S_\sigma(\mathfrak{o}_F)$$

is an isomorphism, and we may view $\Omega_\phi|_{S_\sigma(\mathfrak{o}_E)_\Gamma}$ as a character of $S_\sigma(\mathfrak{o}_F)$ via this isomorphism. Now, since Ω_ϕ was obtained from $\phi|_{\mathcal{W}_E}$ via the local Langlands correspondence for tori, and since $\chi_\phi|_{S_\sigma(\mathfrak{o}_F)}$ was obtained from $\phi|_{\mathcal{W}_E}$ via the local Langlands correspondence for tori, it follows that $\chi_\phi \circ N = \Omega_\phi$ on $S_\sigma(\mathfrak{o}_E)_\Gamma$. We have therefore proven the following lemma.

Lemma 8.1. *The restriction to $S_\sigma(\mathfrak{o}_E)_\Gamma \cong S_\sigma(\mathfrak{o}_F)$ of the genuine character arising from the Gross construction coincides with the character of $S_\sigma(\mathfrak{o}_F)$ that is constructed from ϕ via the construction of DeBacker–Reeder.*

Let $\hat{\mathfrak{T}}(G)$ be the set of all pairs (T, θ), were T is an F-minisotropic maximal torus of G, and $\theta \in \widehat{T(F)}$. There is a natural action of $G(F)$ on $\hat{\mathfrak{T}}(G)$ and a natural notion of stable conjugacy on $\hat{\mathfrak{T}}(G)$. DeBacker and Reeder [11] associate to ϕ a stable conjugacy class $\hat{\mathcal{T}}_{st}$ in $\hat{\mathfrak{T}}(G)$ (see [11, §10.1, §9]). Specifically, for $(T, \theta) \in \hat{\mathcal{T}}_{st}$, the torus T is in the stable class determined by S_σ, and $\theta \in \widehat{T(F)}$ is obtained by transporting χ_ϕ from $S_\sigma(F)$ to $T(F)$ via conjugation.

Choose a set of representatives $\{(T, \theta)\}_{T,\theta}$ for the $G(F)$-orbits contained in $\hat{\mathcal{T}}_{st}$. Then T ranges over a set of representatives for the rational conjugacy classes of tori in the stable class determined by S_σ, and for each such T, θ ranges over a set of representatives for a $\mathscr{W}_F(T)$-orbit in the $W_F(T)$-orbit of characters attached to χ_ϕ. Let $\hat{\mathcal{T}}$ be a rational class in $\hat{\mathcal{T}}_{\mathrm{st}}$ represented by (T, θ). DeBacker and Reeder associate to $\hat{\mathcal{T}}$ a depth-zero supercuspidal representation, which we will denote by $\pi(\hat{\mathcal{T}})$.

Let $\Theta_{\pi(\hat{\mathcal{T}})}$ denote the character of $\pi(\hat{\mathcal{T}})$. For $s \in W_F(T)$, let $s_*\theta(\gamma) := \theta(s^{-1}\gamma s)$. As in [11, §4.4], let $\epsilon(G, T)$ denote $(-1)^{r_1 - r_2}$, where r_1 and r_2 are the respective relative ranks of G and T over F. Let $T(F)_{0,s}$ denote the set of strongly regular topologically semisimple elements of $T(F)$ (see [11, §7]). DeBacker and Reeder prove the following see [11, §10, Thm. 4.5.3].

Theorem 8.2. *For all $\gamma \in T(F)_{0,s}$,*

$$\Theta_{\pi(\hat{\mathcal{T}})}(\gamma) = \epsilon(G, T) \sum_{w \in \mathscr{W}_F(T)} w_*\theta(\gamma).$$

Moreover, the L-packet of G associated to the TRSELP ϕ is $\{\pi(\hat{\mathcal{T}}')\}$, where $\hat{\mathcal{T}}'$ ranges over the set of rational classes in $\hat{\mathcal{T}}_{\mathrm{st}}$.

Remark 8.3. We remark that for the group that we are interested in, $PGSp(4, F)$, it is always the case that if T is unramified elliptic, then $T(\mathfrak{o}_F) \cong T(F)$. In particular, $\hat{H}^0(\Gamma, T(E)) = 1$ and we have an exact sequence

$$1 \to \hat{H}^{-1}(\Gamma, T(E)) \to T(E)_\Gamma \to T(F) \to 1.$$

In addition, $\epsilon(G, T_i) = 1$ for $i = 0, 1, 2$.

Remark 8.4. Let T be a torus in a stable conjugacy class of F-minisotropic maximal tori of G determined by ϕ. Then once one chooses a particular isomorphism $X^*(T) \longrightarrow X_*(\hat{T})$ that implements duality between T and $\hat{T}$, ϕ determines a character ξ of $T(E)_\Gamma$ via the theory of groups of type L. It is easily seen that there exists $g \in G(\overline{F})$ such that $\mathrm{Int}(g)(T) = S$ and $\xi = \Omega_\phi \circ \mathrm{Int}(g)$. Altering the duality isomorphism has the effect of replacing ξ by $w_*\xi = \xi \circ \mathrm{Int}(w^{-1})$ for some $w \in W_F(T)$. Moreover, if $\theta = \chi_\phi \circ \mathrm{Int}(g)$, then (T, θ) is in the stable class $\hat{\mathcal{T}}_{\mathrm{st}}$ determined by ϕ, and moreover, ξ and θ bear the same relationship to each other as χ_ϕ and ξ do in Lemma 8.1. That is $\theta \circ N = \xi$ on $T(\mathfrak{o}_F)$.

Remark 8.5. Let $\hat{\mathcal{T}}_{\mathrm{st}}$ be the stable class determined by ϕ. Let Given T, ξ and θ as in the preceding remark, let $\hat{\mathcal{T}}$ be the rational class of (T, θ) in $\hat{\mathfrak{T}}(G)$. In the case $G = PGSp(4)$, we show in §9 that $\pi(\hat{\mathcal{T}})$ is the unique depth-zero representation satisfying the condition in Conjecture 1.1(1), i.e., $\pi(T, \xi) = \pi(\hat{\mathcal{T}})$. Then for $G = PGSp(4)$ (or in general assuming the previous sentence holds for general G), Conjecture 1.1(2) follows from this fact, Theorem 8.2 and Remark 8.4.

9 Character formulas

9.1 Our conjectural character formula

In this section, we return to the setting in which $G = PGSp(4)$. We define our character formula and prove that it agrees with the character of a unique depth-zero supercuspidal representation on $T(F)_{0,s}$ for an appropriate torus T.

Let ϕ be a TRSELP. If ϕ is of type r^2 (see Definition 7.2), let i be either 0 or 1, and if ϕ is of type r, let $i = 2$. Let ξ be the character of $S_\sigma(E)_\Gamma$ determined by ϕ as in §8. For $i = 0, 1, 2$, it follows from Remark 8.4 that the element g_i from §6 conjugates the character Ω_ϕ to a character ξ_i of $T_i(E)_\Gamma$, and that ξ_i arises from ϕ via the theory of groups of type L, applied to the torus T_i.

As in §6, we transfer the positive roots of the diagonal torus S to the torus T_i by conjugating by the element g_i.

In §6, we described the group $T_i(F)$ of F-rational points via the isomorphism ψ_i. Under these identifications, the two-fold cover $\widetilde{T_i(F)}$ arising from Langlands parameters via the theory of groups of type L is

$$\widetilde{T_i(F)} = \begin{cases} (E_1^* \times E_1^*)/(E_1^* \times N_{E_1/F}(E_1^*)) & \text{if } i = 0, 1, \\ E_2^*/N_{E_2/E_1}(E_2^*) & \text{if } i = 2. \end{cases}$$

Moreover, we have the identification

$$T_i(F) = \begin{cases} \ker(N_{E_1/F}) \times \ker(N_{E_1/F}) & \text{if } i = 0, 1, \\ \ker(N_{E_2/E_1}) & \text{if } i = 2. \end{cases}$$

The covering norm map $N : \widetilde{T_i(F)} \to T_i(F)$ is

$$\begin{array}{rcl} ([w_1], [w_2]) \mapsto & (w_1/\sigma(w_1), w_2/\sigma(w_2)) & \text{if } i = 0, 1, \\ [w] \mapsto & w/\tau^2(w) & \text{if } i = 2. \end{array}$$

We will compute our character formula using these realizations of $\widetilde{T_i(F)}$. In particular, we will need to compute root values on elements of $T_i(F)$. This is done by pulling back via ψ_i^{-1}. In particular, if $z \in \operatorname{im} \psi_i$ and ν is a positive root as above, then $\nu(z)$ is defined by $\nu(\psi_i^{-1}(z))$. In particular, it is readily computed that the values of the positive roots on $T_i(F)$ for $i = 0, 1$ are

$$\alpha((z_1, z_2)) = z_1, \quad \beta((z_1, z_2)) = z_2, \quad (\alpha + \beta)((z_1, z_2)) = z_1 z_2, \quad (2\alpha + \beta)((z_1, z_2)) = z_1^2 z_2,$$

while the values of the positive roots on $T_2(F)$ are

$$\alpha(z) = z, \quad \beta(z) = \tau(z)/z, \quad (\alpha + \beta)(z) = \tau(z), \quad (2\alpha + \beta)(z) = z\tau(z).$$

We now define our Weyl denominator. Recall from §3.2 that in the real case, the Weyl denominator was given by $\Delta^0(t, \Delta^+)\rho(\tilde{t})$, where $t \in T(\mathbb{R})$ and $\tilde{t}$ was a lift of t to $T(\mathbb{R})_\rho$. We now define a p-adic analogue of $\Delta^0(t, \Delta^+)\rho(\tilde{t})$.

For a regular semisimple element $t \in T_i(F)$, note that $\Delta^0(t, \Delta^+)$ takes values in E_i^*, not $\mathbb{C}^*$. Hence we must compose Δ^0 with an appropriate $\mathbb{C}^*$-valued character η which we define later. For the p-adic version of $\rho(\tilde{t})$, we wish to define a complex-valued function on $\widetilde{T_i(F)}$ that will act as a "square root of $\eta \circ (2\rho)$", in analogy with the case of real groups. More precisely, we will define a function η_ρ on $T_i(E)_\Gamma$ that is a canonical square root of $\eta \circ (2\rho) \circ N$, which we will later compute in the separate case T_1 and T_2.

In order to define our character formula, we need to remark that the action of $W_F(T_i)$ on $T_i(E_i)$ descends to one on $T_i(E_i)_{\Gamma_i}$ and $\widetilde{T_i(F)}$.

Remark 9.1. It is easily seen that $(N(G, T_i)/T_i)(F)$ stabilizes $\ker \xi \subset T_i(E_i)$, hence acts on $\widetilde{T_i(F)}$.

We now define a character formula associated to the Langlands parameter ϕ. Abbreviate $\mathscr{W}_F(T_i)$ by $\mathscr{W}_i$ for $i = 0, 1, 2$ We will define η and η_ρ in §9.1.1.

Definition 9.2. Let γ be a regular semisimple element of $T_i(F)$. Define

$$\Theta_\xi(\gamma) := \frac{\sum\limits_{w \in \mathscr{W}_i} w_* \xi(\tilde{\gamma})}{\eta(\Delta^0(\gamma, \Delta^+))\eta_\rho(\tilde{\gamma})}$$

where $\tilde{\gamma}$ is any element of $T_i(E_i)_{\Gamma_i}$ such that $N(\tilde{\gamma}) = \gamma$. Here $w_*\xi(\tilde{\gamma}) := \xi(w^{-1}(\tilde{\gamma}))$ for $w \in \mathscr{W}_i$. (Recall from Lemma 4.2 that $\mathscr{W}_i$ does indeed act on $T(E_i)_\Gamma$.)

A priori, this expression depends on the particular choice of $\tilde{\gamma}$, not just γ. However, we will verify in §9.1.2 and §9.1.3 that it doesn't depend on the choice. We let $D(\tilde{\gamma})$ denote the denominator of this formula, i.e.,

$$D(\tilde{\gamma}) = \eta(\Delta^0(\gamma, \Delta^+))\eta_\rho(\tilde{\gamma}).$$

The next section concerns the character η_ρ appearing in our Weyl denominator. In the two subsequent sections, we construct η_ρ for T_i, and then we prove that Θ_ξ agrees, via stable conjugation by g_i (see §6), with the character of a depth zero supercuspidal representation on $T_i(F)_{0,s}$.

9.1.1 The p-adic Weyl denominator

In §9.1.2 and §9.1.3, for each of the two tori T_i, $i = 0, 1, 2$, we construct a genuine character η_ρ of $\widetilde{T_i(F)}$ such that we have a commutative diagram

$$\begin{array}{ccc} \widetilde{T_i(F)} & \xrightarrow{\eta_\rho} & \mathbb{C}^* \\ \big\downarrow N & & \big\downarrow s \\ T_i(F) & \xrightarrow{\eta \circ (2\rho)} & \mathbb{C}^* \end{array}$$

where $s : \mathbb{C}^* \to \mathbb{C}^*$ is the map $s(z) = z^2$. Equivalently, we will show that there exists a genuine character η_ρ of $\widetilde{T_i(F)}$ whose square is the character $\eta \circ (2\rho) \circ N$. The character η_ρ is unique up to multiplication by a non-genuine character of $\widetilde{T_i(F)}$ of order dividing 2. We will require that η_ρ satisfy the additional condition that its restriction to the image of $T_i(\mathfrak{o}_{E_i})_{\Gamma_i}$ in $\widetilde{T_i(F)}$ be trivial. It is easily seen that this determines η_ρ uniquely (recall that $\widetilde{T_i(F)}$ is a two-fold cover of $T_i(F)$). The character η_ρ constructed in §9.1.2 and §9.1.3 does indeed satisfy this extra condition and is therefore the unique character of $\widetilde{T_i(F)}$ with these properties. By pullback under the canonical projection $T_i(E_1)_{\Gamma_1} \to \widetilde{T_i(F)}$, η_ρ may be viewed as a character of $T_i(E_1)_{\Gamma_1}$.

9.1.2 The tori $T_0(F)$ and $T_1(F)$

We first consider the case in which ϕ is of type r^2 and hence a character formula on $T_i(F)$, where $i = 0$ or 1. Let $\eta = \omega_{E_2/E_1}$. Let $(z_1, z_2) \in T_i(F)$. Then

$$\eta(\Delta^0((z_1, z_2), \Delta^+)) = \eta\left(1 - \frac{1}{z_1}\right)\eta\left(1 - \frac{1}{z_2}\right)\eta\left(1 - \frac{1}{z_1 z_2}\right)\eta\left(1 - \frac{1}{z_1^2 z_2}\right).$$

Note that this equals

$$\eta\left(1 - \frac{\sigma(w_1)}{w_1}\right)\eta\left(1 - \frac{\sigma(w_2)}{w_2}\right)\eta\left(1 - \frac{\sigma(w_1 w_2)}{w_1 w_2}\right)\eta\left(1 - \frac{\sigma(w_1^2 w_2)}{w_1^2 w_2}\right)$$

for any $([w_1], [w_2]) \in T_1(E_1)_\Gamma$ such that $N([w_1], [w_2]) = (z_1, z_2)$.

We now determine the character η_ρ. To calculate η_ρ, we must find a canonical square root of $\eta \circ (2\rho) \circ N$. For $([w_1], [w_2]) \in \widetilde{T_i(F)}$, we have

$$\eta((2\rho)(N([w_1], [w_2]))) = \eta((2\rho)((w_1/\sigma(w_1), w_2/\sigma(w_2)))) =$$

$$\eta\left(\frac{w_1}{\sigma(w_1)}\frac{w_2}{\sigma(w_2)}\frac{w_1 w_2}{\sigma(w_1 w_2)}\frac{w_1^2 w_2}{\sigma(w_1^2 w_2)}\right) = \eta\left(\frac{w_1^4 w_2^3}{\sigma(w_1^4 w_2^3)}\right).$$

Multiplying by $\eta(\sigma(w_2))/\eta(\sigma(w_2))$, and noting that η is trivial on $w_2\sigma(w_2)$, we get

$$\eta(2\rho(N([w_1], [w_2]))) = \eta\left(\frac{w_1^4 w_2^2}{\sigma(w_1^4 w_2^4)}\right).$$

Observe that the character

$$([w_1], [w_2]) \mapsto \eta\left(\frac{w_1^2 w_2}{\sigma(w_1^2 w_2^2)}\right)$$

is a square root of $\eta \circ (2\rho) \circ N$. Moreover, this character is trivial on the image of $T_i(\mathfrak{o}_{E_1})_{\Gamma_1}$ in $\widetilde{T_i(F)}$ and is genuine since it is nontrivial on $([1], [\varpi]) \in \hat{H}^{-1}(\Gamma_1, T_i(E_1))$. Hence this character must equal the character η_ρ from 9.1.1. Therefore, we have that

$$D([w_1], [w_2])$$

$$= \eta\left(1 - \frac{\sigma(w_1)}{w_1}\right)\eta\left(1 - \frac{\sigma(w_2)}{w_2}\right)\eta\left(1 - \frac{\sigma(w_1 w_2)}{w_1 w_2}\right)\eta\left(1 - \frac{\sigma(w_1^2 w_2)}{w_1^2 w_2}\right)\eta\left(\frac{w_1^2 w_2}{\sigma(w_1^2 w_2^2)}\right)$$

$$= \eta\left(\frac{1}{\sigma(w_1)} - \frac{1}{w_1}\right)\eta\left(\frac{1}{\sigma(w_2)} - \frac{1}{w_2}\right)\eta\left(\frac{1}{\sigma(w_1w_2)} - \frac{1}{w_1w_2}\right)\eta\left(w_1^2w_2 - \sigma(w_1^2w_2)\right).$$

Since η is trivial on the norms, we may multiply this whole expression by $\eta(w_1w_2\sigma(w_1w_2))^2 = 1$ to get

$$\eta(w_1 - \sigma(w_1))\eta(w_2 - \sigma(w_2))\eta(w_1w_2 - \sigma(w_1w_2))\eta(w_1^2w_2 - \sigma(w_1^2w_2)).$$

Proposition 9.3. *Let* $\theta = \chi_\phi \circ \mathrm{Int}(g_i)$. *Let* $\hat{\mathcal{T}}$ *be the rational class of* (T_i, θ) *in* $\hat{\mathfrak{T}}(G)$. *Then* $\Theta_\xi(\gamma) = \Theta_{\pi(\hat{\mathcal{T}})}(\gamma)$ *for all* $\gamma \in T_i(F)_{0,s}$.

Proof. Let $\gamma = (z_1, z_2) \in T_i(F)$ Assume that γ is strongly regular and topologically semisimple. The assumption of topological semisimplicity on γ means that the z_i are roots of unity. Moreover, the assumption of strong regularity implies that $z_1 \neq 1, z_2 \neq 1, z_1z_2 \neq 1, z_1^2z_2 \neq 1$. We will compute the terms in Θ_ξ on $T_i(E_1)_{\Gamma_1}$. Note that η_ρ pulls back to a character of $T_i(E_1)_{\Gamma_1}$ via the canonical projection $T_i(E_1)_{\Gamma_1} \to \widetilde{T_i(F)}$. We will denote the latter character by η_ρ as well. Let $\tilde{\gamma} = ([w_1], [w_2]) \in T_i(E_1)_{\Gamma_1} = (E_1^*/N_{E_1/F}(E_1^*)) \times (E_1^*/N_{E_1/F}(E_1^*))$ satisfy $N(\tilde{\gamma}) = \gamma$.

By Remark 8.3, we have that N gives an isomorphism $T_i(\mathfrak{o}_{E_1})_{\Gamma_1} \cong T_i(F)$. Thus, given γ as above, one can choose $\tilde{\gamma}$ to lie in $T_i(\mathfrak{o}_{E_1})_{\Gamma_1}$. We suppose first that this is the case, i.e., that $w_i \in \mathfrak{o}_{E_1}$ for $i = 1, 2$. Then w_1 and w_2 may be chosen to be roots of unity. Plugging in w_1, w_2, we get that $w_1/\sigma(w_1) \neq 1, w_2/\sigma(w_2) \neq 1, (w_1w_2)/(\sigma(w_1w_2)) \neq 1, (w_1^2w_2)/(\sigma(w_1^2w_2)) \neq 1$. Since w_1, w_2 are roots of unity and η is unramified, we therefore get that

$$\eta(w_1 - \sigma(w_1))\eta(w_2 - \sigma(w_2))\eta(w_1w_2 - \sigma(w_1w_2))\eta(w_1^2w_2 - \sigma(w_1^2w_2)) = 1.$$

Now recall from §8 that $\xi = \chi_\phi \circ N$ on $T_1(\mathfrak{o}_{E_1})_{\Gamma_1}$. Therefore,

$$\Theta_\xi(\gamma) = \frac{\sum_{n\in\mathscr{W}_i} n_*\xi([w_1], [w_2])}{D([w_1], [w_2])} = \sum_{n\in\mathscr{W}_i} n_*\xi([w_1], [w_2]) = \Theta_{\pi(T_i,\theta)}(\gamma)$$

for $([w_1], [w_2]) \in T_i(\mathfrak{o}_{E_1})_{\Gamma_1}$ mapping to $\gamma = (z_1, z_2)$ via the norm map.

Now suppose that $w_1, w_2 \in E_1^*$ are arbitrary, i.e., $\tilde{\gamma}$ is an arbitrary element of $T_i(E_1)_{\Gamma_1}$ such that $N(\tilde{\gamma}) = \gamma$. By Remark 8.3, $\tilde{\gamma} = \tilde{\gamma}_0\delta$ for some $\gamma_0 \in T_i(\mathfrak{o}_{E_1})_{\Gamma_1}$ and $\delta \in \hat{H}^{-1}(\Gamma_1, T_i(E_1))$. It suffices to show that $\Theta_\xi(\gamma_0\delta) = \Theta_\xi(\gamma_0)$ for all $\delta \in \hat{H}^{-1}(\Gamma_1, T_i(E_1))$ and strongly regular, topologically semisimple $\gamma_0 \in T_i(\mathfrak{o}_{E_1})_\Gamma$.

First consider $\delta = ([\varpi], [\varpi])$. Note that

$$\begin{aligned} &D([\varpi w_1], [\varpi w_2]) \\ &= \eta(\varpi w_1 - \sigma(\varpi w_1))\eta(\varpi w_2 - \sigma(\varpi w_2))\eta(\varpi w_1\varpi w_2 - \sigma(\varpi w_1\varpi w_2)) \\ &\quad \cdot\eta\left((\varpi w_1)^2\varpi w_2 - \sigma((\varpi w_1)^2\varpi w_2)\right) \\ &= \eta(\varpi)^7 D([w_1], [w_2]) = -D([w_1], [w_2]). \end{aligned} \tag{13}$$

Moreover,

$$\sum_{n\in\mathscr{W}_i} n_*\xi([\varpi w_1], [\varpi w_2]) = \sum_{n\in\mathscr{W}_i} n_*\xi([\varpi], [\varpi]) \cdot n_*\xi([w_1], [w_2]).$$

To simplify this sum, we need to compute the Weyl group $\mathscr{W}_i$ and its action on $T_i(E_1) = E_1^* \times E_1^*$. Since T_i is elliptic, there is a unique maximal compact subgroup K containing $T_i(F)$. (If $i = 0$, K is hyperspecial, while if $i = 1$, K contains a non-hyperspecial maximal parahoric with index 2.) The quotient K/K^+ of K by its pro-unipotent radical is the group of $\mathfrak{f}$-points of a reductive $\mathfrak{f}$-group G. (If $i = 0$, then $\mathsf{G} \cong PGSp(4)$, while if $i = 1$, then $\mathsf{G} \cong \mathsf{H} \rtimes (\mathbb{Z}/2\mathbb{Z})$, where H is semisimple of type $A_1 \times A_1$.) The image of $T_i(F)$ in $\mathsf{G}(\mathfrak{f})$ is the set of $\mathfrak{f}$-points of an elliptic maximal $\mathfrak{f}$-torus $\mathsf{T} \subset \mathsf{G}$. Since r^2 commutes with every element of the Weyl group $W(\mathsf{G}, \mathsf{T})$ of T in G, it follows that every element of the normalizer $N(\mathsf{G}, \mathsf{T})$ of T in G has a representative in $N(\mathsf{G}(\mathfrak{f}), \mathsf{T}(\mathfrak{f}))$. It is easily seen from [10, Lemma 2.2.2] that each such representative lifts to an element of $N(G(F), T_i(F))$. It follows that $\mathscr{W}_i \cong W(\mathsf{G}, \mathsf{T})$, and since $W(G, T_i) \cong W(\mathsf{G}, \mathsf{T})$, we have $\mathscr{W}_i = W(G, T_i)$.

Remark 9.4. Since there are two rational conjugacy classes of maximal F-tori in the stable conjugacy class of T_i and since $\mathscr{W}_i = W_F(G, T_i) = W(G, T_i)$, it follows from §8 that that the L-packet attached to a parameter ϕ of type r^2 has size 2. Moreover, since $|W_F(G, T_i)| = 8$, an elementary computation shows that the number of $W_F(G, T_i)$-orbits of depth-zero regular characters of $T_i(F)$ is $(q^2 - 4q + 3)/8$. From §8, we see that this is the number of distinct L-packets of this form.

Let w_α, w_β be the simple reflections in $W(G, T_i)$ corresponding to the simple roots α and β. Using the isomorphism ψ_i as in Lemma 6.1, we have $w_\nu(w, z) := \phi(w_\nu(\phi^{-1}(w, z)))$ for all $(w, z) \in E_1^* \times E_1^*$ and any root ν. The action of w_α and w_β on $T_i(E_1)$ is given by

$$w_\alpha((x_1, x_2, x_3, x_4)E_1^*) = (x_2, x_1, x_4, x_3)E_1^* \qquad w_\beta((x_1, x_2, x_3, x_4)E_1^*) = (x_1, x_3, x_2, x_4)E_1^*.$$

Thus we have

$$\begin{aligned} w_\alpha(w, z) &= \phi(w_\alpha(w, 1, 1/z, 1/(wz))) = \phi((1, w, 1/(wz), 1/z)) = (1/w, w^2 z) \\ w_\beta(w, z) &= \phi(w_\beta(w, 1, 1/z, 1/(wz))) = \phi((w, 1/z, 1, 1/(wz))) = (wz, 1/z). \end{aligned}$$

Using the above formulas, we get that $w_\alpha([\varpi], [\varpi]) = ([1/\varpi], [\varpi^3])$. But $([1/\varpi], [\varpi^3]) = ([\varpi], [\varpi]) \in E_1/N_{E_1/F}(E_1^*) \times E_1/N_{E_1/F}(E_1^*)$ since $\varpi^2 \in N_{E_1/F}(E_1^*)$. Therefore, $w_\alpha([\varpi], [\varpi]) = ([\varpi], [\varpi])$. Similarly, $w_\alpha([1], [\varpi]) = ([1], [\varpi])$. Moreover, we have $w_\beta([\varpi], [\varpi]) = ([1], [\varpi])$ and, since w_β has order 2, $w_\beta([1], [\varpi]) = ([\varpi], [\varpi])$. It follows that we must have $w_\alpha([\varpi], [1]) = w_\beta([\varpi], [1]) = ([\varpi], [1])$.

Since $\mathscr{W}_i$ is generated by w_α, w_β and since $\xi([\varpi], [\varpi]) = \xi([1], [\varpi]) = -1$, we have

$$\sum_{n \in \mathscr{W}_i} n_* \xi([\varpi], [\varpi]) \cdot n_* \xi([w_1], [w_2]) = - \sum_{n \in \mathscr{W}_i} n_* \xi([w_1], [w_2]).$$

Therefore, by (13),

$$\frac{\sum_{n \in \mathscr{W}_i} n_* \xi([\varpi w_1], [\varpi w_2])}{D([\varpi w_1], [\varpi w_2])} = \frac{\sum_{n \in \mathscr{W}_i} n_* \xi([w_1], [w_2])}{D([w_1], [w_2])}.$$

Now consider the case $\delta = ([1], [\varpi])$. It is easy to see that $D([w_1], [\varpi w_2]) = \eta(\varpi)^3 D([w_1], [w_2])$. Since $\eta(\varpi) = -1$, we have $D([w_1], [\varpi w_2]) = -D([w_1], [w_2])$. Also, the above discussion of the action of $\mathscr{W}_i$ shows that

$$\sum_{n \in \mathscr{W}_i} n_* \xi([1], [\varpi]) \cdot n_* \xi([w_1], [w_2]) = - \sum_{n \in \mathscr{W}_i} n_* \xi([w_1], [w_2]),$$

which handles the element $([1], [\varpi])$.

Now consider $([\varpi], [1])$. It is easy to see that

$$D([\varpi w_1], [w_2]) = \eta(\varpi)^4 D([w_1], [w_2]) = D([w_1], [w_2]).$$

Moreover,

$$\sum_{n\in\mathscr{W}_i} n_*\xi([\varpi w_1],[w_2]) = \sum_{n\in\mathscr{W}_i} n_*\xi([\varpi],[1])\cdot n_*\xi([w_1],[w_2]).$$

Since w_α, w_β both fix $([\varpi],[1])$, and since $\xi([\varpi],[1]) = 1$, we have

$$\frac{\sum_{n\in\mathscr{W}_i} n_*\xi([\varpi w_1],[w_2])}{D([\varpi w_1],[w_2])} = \frac{\sum_{n\in\mathscr{W}_i} n_*\xi([w_1],[w_2])}{D([w_1],[w_2])}.$$

□

9.1.3 The torus $T_2(F)$

We now consider the case in which ϕ is of type r and hence a character formula on $T_2(F)$. Let η be an unramified character of E_2^* of order 4. Let z be an element of $\ker(N_{E_2/E_1})$, which we identify with $T_2(F)$ as in Lemma 6.7. Then

$$\eta(\Delta^0(z,\Delta^+)) = \eta\left(1-\frac{1}{z}\right)\eta\left(1-\frac{z}{\tau(z)}\right)\eta\left(1-\frac{1}{\tau(z)}\right)\eta\left(1-\frac{1}{z\tau(z)}\right). \tag{14}$$

Let $[w] \in E_2^*/N_{E_2/E_1}(E_2^*) = \widetilde{T_2(F)} = T_2(E_2)_{\Gamma_2}$ satisfy $N(w) = w/\tau^2(w) = z$. Then (14) equals

$$\eta\left(1-\frac{\tau^2(w)}{w}\right)\eta\left(1-\frac{w\tau^3(w)}{\tau(w)\tau^2(w)}\right)\eta\left(1-\frac{\tau^3(w)}{\tau(w)}\right)\eta\left(1-\frac{\tau^2(w)\tau^3(w)}{w\tau(w)}\right).$$

We now wish to define the function η_ρ on $T_2(E_2)_{\Gamma_2}$ as we did for $T_1(E_1)_{\Gamma_1}$. Again, we first compute $\eta\circ(2\rho)\circ N$. For $[w] \in T_2(E)_\Gamma$, let $z = N([w]) = w/\tau^2(w)$. Then $\eta((2\rho)(z)) = \eta(z\tau(z)^3)$ so

$$\eta((2\rho)(N([w]))) = \eta\left(\frac{w\tau(w)^3}{\tau^2(w)\tau^3(w)^3}\right).$$

If we multiply this by $\eta(w\tau(w))/\eta(w\tau(w))$, we get

$$\eta((2\rho)(N([w])) = \eta\left(\frac{w^2\tau(w)^4}{w\tau(w)\tau^2(w)\tau^3(w)\tau^3(w)^2}\right) = \eta\left(\frac{w^2\tau(w)^4}{\tau^3(w)^2}\right)$$

since η is trivial on $N_{E_2/F}(E_2^*)$. Thus the character

$$w \mapsto \eta\left(\frac{w\tau(w)^2}{\tau^3(w)}\right)$$

is a square root of $\eta\circ(2\rho)\circ N$. This character is trivial on the image of $T_2(\mathfrak{o}_{E_2})_{\Gamma_2}$ and nontrivial on $\hat{H}^{-1}(\Gamma_2, T_2(E_2))$, hence must equal η_ρ (see 9.1.1).

Therefore, we get

$$\begin{aligned} D([w]) &= \eta(\Delta^0(N([w]),\Delta^+))\eta_\rho([w]) \\ &= \eta\left(1-\frac{\tau^2(w)}{w}\right)\eta\left(1-\frac{w\tau^3(w)}{\tau(w)\tau^2(w)}\right)\eta\left(1-\frac{\tau^3(w)}{\tau(w)}\right)\eta\left(1-\frac{\tau^2(w)\tau^3(w)}{w\tau(w)}\right)\eta\left(\frac{w\tau(w)^2}{\tau^3(w)}\right) \\ &= \eta\left(w-\tau^2(w)\right)\eta\left(\tau(w)-\frac{w\tau^3(w)}{\tau^2(w)}\right)\eta\left(\tau(w)-\tau^3(w)\right)\eta\left(\frac{1}{\tau^3(w)}-\frac{\tau^2(w)}{w\tau(w)}\right) \end{aligned} \tag{15}$$

Since η is trivial on $N_{E_2/F}(E_2^*)$, we have $\eta(w\tau(w)\tau^2(w)\tau^3(w)) = 1$. Thus, multiplying (15) by $\eta(w\tau(w)\tau^2(w)\tau^3(w))$, we find that

$$D([w]) = \eta(w-\tau^2(w))\cdot\eta(\tau(w)\tau^2(w)-w\tau^3(w))\cdot\eta(\tau(w)-\tau^3(w))\cdot\eta(w\tau(w)-\tau^2(w)\tau^3(w)). \quad (16)$$

Proposition 9.5. *Let $\theta = \chi_\phi \circ \mathrm{Int}(g_2)$. Let $\hat{\mathcal{T}}$ be the rational class of (T_2,θ) in $\hat{\mathfrak{T}}(G)$. Then $\Theta_\xi(\gamma) = \Theta_{\pi(\hat{\mathcal{T}})}(\gamma)$ for all $\gamma \in T_2(F)_{0,s}$.*

Proof. Let $\gamma \in T_2(F)$ correspond to the element $z \in \ker(N_{E_2/E_1})$ via our realization of $T_2(E_2)$ in §6.2. Suppose that γ is strongly regular and topologically semisimple. Then z is a root of unity, and moreover, $z \neq 1$, $z \neq \tau(z)$, and $z \neq \tau(z)^{-1}$. Let $\tilde{\gamma} \in T_2(E_2)_{\Gamma_2}$ satisfy $N(\tilde{\gamma}) = \gamma$. Then $\tilde{\gamma}$ corresponds to an element $[w] \in E_2^*/N_{E_2/E_1}(E_2^*)$ (as in Lemma 6.7) such that $w/\tau^2(w) = z$. The strong regularity of γ implies that $w \neq \tau^2(w)$, $w\tau^3(w) \neq \tau(w)\tau^2(w)$, and $w\tau(w) \neq \tau^2(w)\tau^3(w)$.

As in the case of the torus T_1, by Remark 8.3, N gives an isomorphism $T_2(\mathfrak{o}_{E_2})_{\Gamma_2} \cong T_2(F)$. Thus, given γ as above, one can choose $\tilde{\gamma}$ to lie in $T_2(\mathfrak{o}_{E_2})_{\Gamma_2}$. Suppose first that this is the case, i.e., that $w \in \mathfrak{o}_{E_2}$. Then w may be chosen to be a root of unity. It follows from this, the fact that η is unramified, and (16) that $D([w]) = 1$. Since $\xi = \chi_\phi \circ N$ on $T_2(\mathfrak{o}_{E_2})_{\Gamma_2}$ (see §8), we obtain

$$\Theta_\xi(\gamma) = \frac{\sum_{n\in\mathscr{W}_2} n_*\xi([w])}{D([w])} = \sum_{n\in\mathscr{W}_2} n_*\xi([w]) = \Theta_{\pi(T_2,\theta)}(\gamma).$$

Now suppose that $w \in E_2^*$ is arbitrary, i.e., $\tilde{\gamma}$ is an arbitrary element of $T_2(E_2)_{\Gamma_2}$ such that $N(\tilde{\gamma}) = \gamma$. As in §9.1.2, it suffices to show that $\Theta_\xi(\gamma_0\delta) = \Theta_\xi(\gamma_0)$ for all $\delta \in \hat{H}^{-1}(\Gamma_2, T_2(E_2))$ and strongly regular, topologically semisimple $\gamma_0 \in T_2(\mathfrak{o}_{E_2})_{\Gamma_2}$. From Lemma 6.6, it follows that the nontrivial element δ of $\hat{H}^{-1}(\Gamma_2, T_2(E_2)) \subset T_2(E_2)_{\Gamma_2}$ is $[\varpi]$.

First note that

$$\begin{aligned} & D([\varpi w]) \\ = \ & \eta(\varpi w - \tau^2(\varpi w))\eta(\tau(\varpi w)\tau^2(\varpi w) - \varpi w\tau^3(\varpi w))\eta(\tau(\varpi w)-\tau^3(\varpi w)) \quad (17) \\ & \cdot\, \eta(\varpi w\tau(\varpi w) - \tau^2(\varpi w)\tau^3(\varpi w)) \\ = \ & \eta(\varpi)^6\eta(w-\tau^2(w))\eta(\tau(w)\tau^2(w)-w\tau^3(w))\eta(\tau(w)-\tau^3(w))\eta(w\tau(w)-\tau^2(w)\tau^3(w)) \\ = \ & \eta(\varpi)^6 D([w]). \end{aligned}$$

This is equal to $-D([w])$ since $\eta(\varpi)$ has order 4.

To simplify the numerator of $\Theta_\xi([\varpi w])$, we compute the Weyl group action on $T_2(E_2) = E_2^* \times E_2^*$. Let w_α, w_β be the simple reflections in the Weyl group $W(G,T_2)$ of $PGSp(4)$ with respect to T_2 corresponding to the simple roots α and β. Under the identification φ in Lemma 6.4, we have

$$\begin{aligned} w_\alpha(w,z) &= \varphi(w_\alpha(wz,z,w,1)) = \varphi(z,wz,1,w) = (1/w,z) \\ w_\beta(w,z) &= \varphi(w_\beta(wz,z,w,1)) = \varphi((wz,w,z,1)) = (z,w). \end{aligned}$$

Recall from Lemma 6.4, that $\tau(w,z) = (\tau(z)^{-1},\tau(w))$. One easily checks that the subgroup $W_F(G,T_2)$ of elements of $W(G,T_2)$ which commute with the action of $\mathrm{Gal}(E_2/F)$ is precisely $\langle w_\alpha w_\beta\rangle$. Moreover, as in §9.1.2, one can show that each element of $W_F(G,T_2)$ has a representative in $N(G,T_2)(F)$, so that $\mathscr{W}_2 = W_F(G,T_2)$.

Remark 9.6. Since the rational conjugacy class of T_2 is stable, and since $\mathscr{W}_2 = W_F(G, T_2)$, it follows from §8 that that the L-packet attached to a parameter ϕ of type r is a singleton. Since $|W_F(G, T_i)| = 4$, an elementary computation shows that the number of $W_F(G, T_2)$-orbits of depth-zero regular characters of $T_i(F)$ is $(q^2 - 1)/4$. From §8, this is the number of distinct L-packets of this form.

Now the element $[\varpi]$ of $E_2^*/N_{E_2/E_1}(E_2^*)$ corresponds to the coset in $T_2(E_2)_{\Gamma_2}$ represented by the element $(\varpi, 1)$ via the identification φ in Lemma 6.4. But $(w_\alpha w_\beta)(\varpi, 1) = (1, \varpi)$, which corresponds to the element $[\varpi^{-1}] = [\varpi] \in E_2^*/N_{E_2/E_1}(E_2^*)$. If follows that $(N(G, T_2)/T_2)(F)$ acts trivially on $\hat{H}^{-1}(\Gamma_2, T_2(E_2))$. Therefore,

$$\sum_{n \in \mathscr{W}_2} n_*\xi([\varpi w]) = \sum_{n \in \mathscr{W}_2} n_*\xi([\varpi]) \cdot n_*\xi([w]) = \sum_{n \in \mathscr{W}_2} \xi([\varpi]) n_*\xi([w]) = -\sum_{n \in \mathscr{W}_2} n_*\xi([w]),$$

since $\xi([\varpi]) = -1$. Therefore, we get

$$\frac{\sum_{n \in \mathscr{W}_2} n_*\xi([\varpi w])}{D([\varpi w])} = \frac{-\sum_{n \in \mathscr{W}_2} n_*\xi([w])}{-D([w])} = \frac{\sum_{n \in \mathscr{W}_2} n_*\xi([w])}{D([w])}$$

and we are done. □

Remark 9.7. We note that our definition of Θ_ξ involved a choice of positive roots Δ^+. We are now in a position to analyze the dependence of this function on Δ^+. Note that the numerator is independent of Δ^+. So it suffices to analyze how $D(\tilde{\gamma})$ ($\tilde{\gamma} \in \widetilde{T_i(F)}$) depends on Δ^+. Selecting a different set of positive roots is equivalent to replacing the conjugating element g_i of §6 by $g_i n$, where $n \in N(G(E_i), T_i(E_i))$. If $n \in N(G(E_i), T_i(F))$, then the results of this and previous sections still hold, and their proofs are unchanged. However, if $n \notin N(G(E_i), T_i(F))$, then in our explicit realization of $T(E_i)$ from §6, the action of Γ_i will be modified slightly. Nevertheless, although the proofs of a number of results must be slightly modified in this case, the results of this section remain true. In particular, the functions Θ_ξ do not depend on the choice of positive roots.

9.2 Uniqueness of restrictions of characters

Let G be an unramified connected reductive F-group. In this section, we show that a depth-zero supercuspidal character of G that comes from the elliptic maximal F-torus T via the construction of [11] is uniquely determined by its restriction to $T(F)_{0,s}$.

Let π be a depth-zero supercuspidal representation of $G(F)$ associated to an elliptic maximal F-torus T as in [11]. We show that if the character of π coincides on $Z(F)T(F)_{0,s}$ with that of another depth-zero supercuspidal representation of $G(F)$, then these representations are equivalent. We do this first when both representations arise from the same elliptic maximal F-torus T of G.

Lemma 9.8. *Let Ξ be a nonempty set of depth-zero characters of $T(F)$, and for each $\xi \in \Xi$, let $a_\xi = \pm 1$. Then for q sufficiently large, the function*

$$f = \sum_{\xi \in \Xi} a_\xi \xi, .$$

is not identically zero on $Z(F)T(F)_{0,s}$.

Proof. Suppose to the contrary that f vanishes on $Z(F)T(F)_{0,s}$. The group $\mathbb{T}^{\Phi_\sigma} \subset \mathbb{T}$ is the quotient of the maximal compact subgroup $T(\mathfrak{o}_F)$ of $T(F)$ by its pro-unipotent radical. Since the characters in Ξ have depth-zero, f can be viewed as a function on $\mathbb{T}^{\Phi_\sigma}$. Thus, if $\chi \in \Xi$, $|\langle f, \chi\rangle| = |a_\chi| = 1$, where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian inner product on the space of complex-valued functions on $\mathbb{T}^{\Phi_\sigma}$.

Let Y be the complement of the image of $T(F)_{0,s} \cap T(\mathfrak{o}_F)$ in $\mathbb{T}^{\Phi_\sigma}$. Then as a function on $\mathbb{T}^{\Phi_\sigma}$, f vanishes off Y. Thus

$$1 = |\langle f, \chi\rangle| = \frac{1}{|\mathbb{T}^{\Phi_\sigma}|}\left|\sum_{y\in Y}\sum_{\xi\in\Xi} a_\xi \xi(y)\cdot\chi^{-1}(y)\right| \leq \frac{|Y||\Xi|}{|\mathbb{T}^{\Phi_\sigma}|}.$$

It follows that

$$|Y|/|\mathbb{T}^{\Phi_\sigma}| \geq 1/|\Xi|. \tag{18}$$

It is easily seen that the absolute Weyl group $W(G,T)$ acts naturally on $\mathbb{T}$ and that Y lies in the union $\mathbb{Y}$ of the fixed-point subgroups $\mathbb{T}^w$ of $\mathbb{T}$ as w ranges over the nontrivial elements of $W(G,T)$. Suppose $y \in Y$ lies in $\mathbb{T}^w$ for some $w \in W(G,T)$. Then $y \in \mathbb{T}^{\Phi_\sigma^n(w)}$ for all $n \in \mathbb{Z}$. Thus y is contained in the $\mathfrak{f}$-subgroup $\bigcap_{n\in\mathbb{Z}} \mathbb{T}^{\Phi_\sigma^n(w)}$ of $\mathbb{T}$. Let V denote the set $\langle\Phi_\sigma\rangle\backslash(W(G,T) - \{1\})$ of orbits of $\langle\Phi_\sigma\rangle$ on $W(G,T) - \{1\}$. Then the preceding discussion shows that for $\mathcal{O} \in V$, $\mathbb{T}^{\mathcal{O}} := \bigcap_{w\in\mathcal{O}} \mathbb{T}^w$ is a $\mathfrak{f}$-subgroup of $\mathbb{T}$ and that

$$Y \subset \bigcup_{\mathcal{O}\in V} \left(\mathbb{T}^{\mathcal{O}}\right)^{\Phi_\sigma}.$$

Note that $\mathbb{T}^{\mathcal{O}}$ is not necessarily connected. However, it is clear that there exists a positive integer C, depending only on the algebraic closure $\bar{\mathfrak{f}}$ of $\mathfrak{f}$, such that $|\mathbb{T}^{\mathcal{O}}/\left(\mathbb{T}^{\mathcal{O}}\right)^\circ| \leq C$ for all $\mathcal{O}$. Then Lang's theorem, together with standard results on Galois cohomology, imply that

$$\left|H^1(\langle\Phi_\sigma\rangle, \mathbb{T}^{\mathcal{O}})\right| = \left|H^1(\langle\Phi_\sigma\rangle, \mathbb{T}^{\mathcal{O}}/\left(\mathbb{T}^{\mathcal{O}}\right)^\circ\right| \leq C$$

and hence that

$$\left|\left(\mathbb{T}/\mathbb{T}^{\mathcal{O}}\right)^{\Phi_\sigma}\right| \leq C\left|\mathbb{T}^{\Phi_\sigma}/\left(\mathbb{T}^{\mathcal{O}}\right)^{\Phi_\sigma}\right|.$$

Thus

$$\frac{|Y|}{|\mathbb{T}^{\Phi_\sigma}|} \leq \sum_{\mathcal{O}\in V}\frac{|(\mathbb{T}^{\mathcal{O}})^{\Phi_\sigma}|}{|\mathbb{T}^{\Phi_\sigma}|} = \sum_{\mathcal{O}\in V}\frac{1}{|\mathbb{T}^{\Phi_\sigma}/(\mathbb{T}^{\mathcal{O}})^{\Phi_\sigma}|} \leq \sum_{\mathcal{O}\in V}\frac{C}{|(\mathbb{T}/\mathbb{T}^{\mathcal{O}})^{\Phi_\sigma}|}. \tag{19}$$

Note that for each $\mathcal{O} \in V$, $\mathbb{T}/\mathbb{T}^{\mathcal{O}}$ is a positive-dimensional $\mathfrak{f}$-torus. It follows from [19, §9.1] that $\left|(\mathbb{T}/\mathbb{T}^{\mathcal{O}})^{\Phi_\sigma}\right| \geq q-1$. Combining this with (19), we get that

$$\frac{|Y|}{|\mathbb{T}^{\Phi_\sigma}|} \leq \frac{C|V|}{q-1},$$

and hence by (18),

$$q \leq C|V||\Xi| + 1 \leq C(|W(G,T)|-1)|\Xi| + 1,$$

and the result follows. □

Proposition 9.9. *Let χ_1, χ_2 be depth-zero characters of $T(F)$ that are in general position with respect to the action of $W_F(T)$. Suppose*

$$\sum_{n \in \mathscr{W}_F(T)} n_* \chi_1(z\gamma_0) = \sum_{n \in \mathscr{W}_F(T)} n_* \chi_2(z\gamma_0)$$

for all strongly regular, topologically semisimple elements $\gamma_0 \in T(F)_{0,s}$ and for all elements $z \in Z(F)$. If q is sufficiently large, then $\chi_1 = n'_* \chi_2$ for some $n' \in \mathscr{W}_F(T)$.

Proof. Consider the function

$$f = \sum_{n \in \mathscr{W}_F(T)} n_* \chi_1 - \sum_{n \in \mathscr{W}_F(T)} n_* \chi_2$$

on $T(F)$. This function vanishes on $Z(F)T(F)_{0,s}$ by assumption. It follows from Lemma 9.8 that for q sufficiently large, when f is expressed as a linear combination of distinct characters, the coefficients cannot all be ± 1. This, together with the fact that χ_1 and χ_2 are in general position, implies that $\chi_1 = n'_* \chi_2$ for some $n' \in \mathscr{W}_F(T)$, as desired. $\square$

Remark 9.10. In the case of $G = PGSp(4)$, one can calculate that the inequality (18) (with $|\Xi| = 2|W|$ as in Proposition 9.9) holds for T_1 when $q > 89$ and for T_2 when $q > 3$.

Remark 9.11. Proposition 9.9 and Theorem 8.2 together imply that if π is a depth-zero supercuspidal representation of $G(F)$ associated to the elliptic maximal F-torus T, then the values of its character on $Z(F)T(F)_{0,s}$ distinguish it from all other such representations associated to T.

Proof of Theorem 1.2. Now let T' be an elliptic maximal F-torus of G that is not F-conjugate to T. By [11, §10.1], the character of a depth zero supercuspidal representation coming from T' vanishes completely on $Z(F)T(F)_{0,s}$. The reason is that if $\gamma_0 \in T(F)_{0,s}$, then γ_0 is not contained in any F-conjugate of T'. We have shown in Lemma 9.8 that the character of any depth zero supercuspidal representation coming from T doesn't vanish on all of $Z(F)T(F)_{0,s}$. Therefore, the character of the above supercuspidal representation π associated to T cannot agree on $Z(F)T(F)_{0,s}$ with the character of a depth-zero supercuspidal representation associated to T'. As a result of this fact, together with Remarks 9.11 and 8.5, and Propositions 9.3 and 9.5, we have now proved Theorem 1.2.

Remark 9.12. There is one minor point here to be stated. We do not claim that there are no other supercuspidal representations outside of [11] of $G(F)$ whose characters agree with Θ_ξ on $Z(F)T(F)_{0,s}$, even though we expect this to be true.

References

[1] Jeffrey Adams, *Extensions of tori in* SL(2). Pacific J. Math. 200 (2001), no. 2, 257–271.

[2] Jeffrey Adams and David Vogan *L-Groups, Projective Representations, and the Langlands Classification.* Amer. Journal of Math. 113 (1991), 45-138.

[3] Jeffrey Adams and David Vogan *The Contragredient,* preprint. arXiv:1201.0496, http://arxiv.org/pdf/1201.0496.pdf

[4] Jeffrey Adler, *Refined anisotropic K-types and supercuspidal representations*, Pacific J. Math., 185 (1998), pp. 1-32.

[5] Moshe Adrian *On the Local Langlands Correspondences of DeBacker/Reeder and Reeder for $GL(\ell, F)$, where ℓ is prime*, Pacific Journal of Mathematics 255-2 (2012), 257–280.

[6] Moshe Adrian *A New Construction of the Local Langlands Correspondence for $GL(n, F)$, n a prime*, Ph.D. Thesis.

[7] M. Adrian and D. Roe, *Rectifiers and the local Langlands Correspondence: the unramified case*, preprint. arXiv:1307.0469, http://arxiv.org/pdf/1307.0469.pdf

[8] Colin Bushnell and Guy Henniart, *The essentially tame local Langlands correspondence, I*, J. Amer. Math. Soc. 18 (2005), no. 3, 685–710.

[9] Roger Carter, *Simple Groups of Lie Type*, John Wiley and Sons Inc, 1993.

[10] Stephen DeBacker, *Parametrizing conjugacy classes of maximal unramified tori via Bruhat-Tits Theory.* Michigan Math. J., 54 (2006), no. 1, 157–178.

[11] Stephen DeBacker and Mark Reeder, *Depth-zero supercuspidal L-packets and their stability.* Ann. of Math. (2) 169 (2009), no. 3, 795–901.

[12] Benedict Gross, *Groups of type L*, unpublished notes.

[13] Benedict Gross and Mark Reeder, *Arithmetic invariants of discrete Langlands parameters.* Duke Math. Journal, 154, (2010), 431-508.

[14] Tasho Kaletha, *Simple Wild L-packets*, preprint.

[15] David Kazhdan and Yakov Varshavsky, *Endoscopic decomposition of certain depth zero representations.* Progr. Math., 243, Birkhauser, Boston, MA, 2006.

[16] Lawrence Morris, *Some tamely ramified supercuspidal representations of symplectic groups.* Proc. London Math. Soc. (3) 63 (1991), no. 3, 519–551.

[17] Mark Reeder, *Supercuspidal L-packets of positive depth and twisted Coxeter elements*, J. Reine Angew. Math. 620 (2008), 1-33.

[18] Loren Spice, *Supercuspidal characters of SL_ℓ over a p-adic field, ℓ a prime.* Amer. J. Math. 121 no. 1, 51100.

[19] Valentin Voskresenskii, *Algebraic Groups and their Birational Invariants*, American Mathematical Society, Providence, RI, 1998.