The Substitution Rule (§4.5)

Recall the chain rule:

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx} f(u) = f'(u) \cdot \frac{du}{dx}$$

The substitution Rule:

If $u = g(x)$ is a differentiable function, then

$$\int f(u) \cdot \frac{du}{dx}\, dx = \int f(u) \cdot du$$

Ex. Find $\int x^2 \sin(x^3)\, dx$

$f(x) = \sin(x)$

$u = x^3$

$\frac{du}{dx} = 3x^2$

$$\int x^2 \sin(x^3) dx = \frac{1}{3} \int 3x^2 \sin(x^3)\, dx$$

$$= \frac{1}{3} \int \underbrace{\sin(x^3)}_{f(u)} \cdot \underbrace{3x^2}_{\frac{du}{dx}} \cdot dx$$

$$= \frac{1}{3} \int \sin(u)\, du$$

$$= -\frac{1}{3} \cos(u) + C$$

$$= -\frac{1}{3} \cos(x^3) + C$$

Check:

$$\frac{d}{dx}\left(-\frac{1}{3} \cos(x^3)\right)$$

$$= \frac{1}{3} \sin(x^3) \cdot 3x^2$$

$$= x^2 \sin(x^3) \checkmark$$

Substitution Rule let's you treat $\frac{du}{dx}$ as a fraction

Again w/ convenient notation:

$$\int x^2 \sin(x^3)\, dx$$

$u = x^3$

$du = 3x^2\, dx \Rightarrow dx = \frac{du}{3x^2}$

$$\int x^2 \sin(x^3)\, dx = \int x^2 \sin(u) \cdot \frac{du}{3x^2}$$

$$= \frac{1}{3} \int \sin(u)\, du$$

$$= -\frac{1}{3} \cos(u) + C$$

$$= -\frac{1}{3} \cos(x^3) + C$$

Example

$$\int x\sqrt{x^2-1}\, dx = \int x\sqrt{u}\, \frac{du}{2x}$$

$u = x^2 - 1$

$du = 2x\, dx$

$$= \frac{1}{2} \int \sqrt{u}\, du$$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot u^{3/2} + C$$

$$= \frac{1}{3} (x^2-1)^{3/2} + C$$

Ex

$$\int \sqrt{5x+2}\, dx = \int \sqrt{u}\, \frac{du}{5}$$

$u = 5x+2$

$du = 5\, dx$

$$= \frac{1}{5} \int u^{1/2}\, du$$

$$= \frac{1}{5} \cdot \frac{2}{3} \cdot u^{3/2} + C$$

$$= \frac{2}{15} (5x+2)^{3/2} + C$$

Ex $\int \frac{\cos(t)}{\sin^2(t)}\,dt = \int \frac{du}{u^2} = -\frac{1}{u} + C$

$= -\frac{1}{\sin(t)} + C$

$u = \sin(t)$

$du = \cos(t)\,dt$

Substitution Rule For Definite Integrals

Example $\int_1^2 \sqrt{x-1}\,dx = \int_0^0 \sqrt{u}\,du$

$u = x-1$

$du = dx$

The limits have to change to match the new variable u.

For the lower limit, when $x=1$, $u=0$

" " upper " , " $x=2$, $u=1$

So, $\int_1^2 \sqrt{x-1}\,dx = \int_0^1 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}$.

Substitution rule for definite integrals

If g' is continuous on $[a,b]$ and f is continuous in the range of $u=g(x)$, then

$$\int_a^b f(u)\cdot\frac{du}{dx}\cdot dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Ex $\displaystyle\int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\, d\theta = \int_1^{\frac{\sqrt{2}}{2}} -\frac{1}{u^2}\, du = \frac{1}{u}\Big|_1^{\frac{\sqrt{2}}{2}}$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$u(0) = 1$

$u(\pi/4) = \frac{\sqrt{2}}{2}$

$= \frac{2}{\sqrt{2}} - 1$

$= \sqrt{2} - 1$

Symmetry

Suppose f is continuous on $[-a, a]$.

a) If f is even $(f(-x) = f(x))$, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$

b) If f is odd $(f(-x) = -f(x))$, then $\int_{-a}^{a} f(x)\, dx = 0$.

Proof $\displaystyle\int_{-a}^{a} f(x)\, dx = \boxed{\int_{-a}^{0} f(x)\, dx} + \int_0^a f(x)\, dx = \int_0^a f(-x)\, dx + \int_0^a f(x)\, dx$

$u = -x$

$du = -dx$

$u(-a) = a$

$u(0) = 0$

$\displaystyle\int_{-a}^{0} f(x)\, dx = \int_a^0 -f(-u)\, du$

$\displaystyle = \int_0^a f(-u)\, du$

$\displaystyle = \int_0^a f(-x)\, dx$

a) $f(-x) = f(x) \Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

b) $f(-x) = -f(x) \Rightarrow \int_{-a}^{a} f(x)\,dx = \int_0^a -f(x)\,dx + \int_0^a f(x)\,dx = 0.$ $\square$

Ex, $\displaystyle\int_{-\pi/2}^{\pi/2} \sin(\theta)\,d\theta = 0$, $\displaystyle\int_{-\pi/2}^{\pi/2} \cos\theta\,d\theta = 2\int_0^{\pi/2} \cos\theta\,d\theta$

$$= 2\sin\theta\Big|_0^{\pi/2}$$

$$= 2$$