Calculus of inverse functions

Theorem If f is one-to-one and continuous, then its inverse f^{-1} is continuous.

Theorem If f is one-to-one and differentiable, then f^{-1} is differentiable at $x=a$ if $f'(f^{-1}(a)) \neq 0$,

Furthermore $(f^{-1})'(a) = \dfrac{1}{f'(f^{-1}(a))}$.

If we assume that f^{-1} is differentiable, then we can see this using implicit differentiation:

$$f(f^{-1}(x)) = x$$

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x$$

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$

$$\Rightarrow (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Ex Let $f(x) = \sin^2 x + 4x + 1$.

a) Show that f has an inverse

b) Find $(f^{-1})'(1)$

a) $f'(x) = 2\sin x \cos x + 4$

$-2 \le 2\sin x \cos x \le 2$, so $f'(x) > 0$ for every x

$\Rightarrow f(x)$ is increasing

$\Rightarrow f(x)$ is one-to-one

$\Rightarrow f(x)$ has an inverse.

b) $f(0) = 1 \Rightarrow f^{-1}(1) = 0$

Now, $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))}$

$= \dfrac{1}{f'(0)}$

$= \dfrac{1}{4}$

The Natural Logarithm (§5.2)

Def The natural logarithmic function is the function defined by $\ln x = \int_1^x \frac{1}{t}\,dt$ for $x>0$.

So, by definition and FTC I, $\ln x$ is the antiderivative $F(x)$ of $f(x) = \frac{1}{x}$ such that $F(1) = 0$.

In other words, $\frac{d}{dx} \ln x = \frac{1}{x}$ and $\ln(1) = 0$.

Ex. Approximate $\ln(2)$:

$$\ln(2) = \int_1^2 \frac{1}{t}\,dt$$

We approximate w/ R_n. Let's chose $n=4$

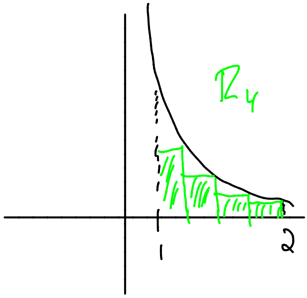
$$\Delta x = \frac{1}{4}$$

$$R_4 = \frac{1}{4}\left(\frac{1}{5/4} + \frac{1}{6/4} + \frac{1}{7/4} + \frac{1}{8/4} \right)$$

$$= \frac{1}{4}\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2} \right)$$

$$= \frac{533}{840}$$

$$\approx 0.6345 \implies \text{so } \ln(2) \approx 0.6345$$



while $\ln(2) = 0.6931\ldots$ (Not the best approximation, but it would better better w/ larger n)

Laws of Logarithms

If x and y are positive numbers and r is a rational number, then

1) $\ln(xy) = \ln x + \ln y$

2) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

3) $\ln(x^r) = r \ln x$

Let's prove 1):

Let $f(x) = \ln(xy)$

So, $\frac{df}{dx} = \frac{d}{dx}\ln(xy) = \frac{1}{xy} \cdot \frac{d}{dx}(xy) = \frac{1}{xy} \cdot y = \frac{1}{x}$

$\Rightarrow$ $\ln(x)$ and $\ln(xy)$ have the same derivative, so they must differ by a constant:

$$\ln(xy) = \ln x + C$$

Setting $x=1$ gives $\ln(y) = C$

$\Rightarrow$ $\ln(xy) = \ln x + \ln y$ ✓

2) and 3) have similar proofs.

Ex. Expand the expression $\ln\left(\frac{x^2\cos^3 x}{x^3+1}\right)$

$$\ln\left(\frac{x^2\cos^3 x}{x^3+1}\right) = \ln(x^2) + \ln(\cos^3 x) - \ln(x^3+1)$$

$$= 2\ln x + 3\ln(\cos x) - \ln(x^3+1)$$

Ex. Express $\frac{1}{2}\ln 4 + 2\ln 3 - \ln 5$ as a single logarithm.

$$\frac{1}{2}\ln 4 = \ln 4^{1/2} = \ln 2$$

$$2\ln 3 = \ln 3^2 = \ln 9$$

So, $\frac{1}{2}\ln 4 + 2\ln 3 - \ln 5 = \ln 2 + \ln 9 - \ln 5$

$$= \ln(2\cdot 9) - \ln 5$$

$$= \ln 18 - \ln 5$$

$$= \ln\left(\frac{18}{5}\right)$$