Recall, by definition, $\frac{d}{dx} \ln x = \frac{1}{x}$

Ex Find $\frac{d}{dx} \ln(\sin x)$

We have to use the chain rule $\left( \frac{d}{dx} f(u(x)) = f'(u(x)) \cdot \frac{du}{dx} \right)$

$$\frac{d}{dx} \ln(\sin x) = \frac{1}{\sin x} \cdot \frac{d}{dx} \sin x = \frac{1}{\sin x} \cdot \cos x$$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x$$

Ex. Find $f'(x)$ if $f(x) = \ln|x|$

$$f(x) = \begin{cases} \ln x & x > 0 \\ \ln(-x) & x < 0 \end{cases}$$

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot -1 = \frac{1}{x}$$

$\Rightarrow f'(x) = \frac{1}{x}$ for $x \neq 0$.

We just saw that $\boxed{\frac{d}{dx} \ln|x| = \frac{1}{x}}$

So, we also have

$$\boxed{\int \frac{1}{x}\, dx = \ln|x| + C}$$

Ex $\int \frac{x^2}{x^3+1}\, dx = \int \frac{x^2}{u} \frac{du}{3x^2} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C$

$= \frac{1}{3}\ln|x^3+1| + C$

$u = x^3+1$

$\frac{du}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

Ex $\int \frac{\sin(\ln x)}{x}\, dx = \int \sin(u)\, du = -\cos(u) + C$

$= -\cos(\ln x) + C$

$u = \ln x$

$du = \frac{1}{x}\, dx$

Ex $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx = -\int \frac{1}{u}\, du = -\ln|u| + C$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$= -\ln|\cos x| + C$

$= \ln|(\cos x)^{-1}| + C$

$= \ln|\sec x| + C$

$$\Rightarrow \boxed{\int \tan x\, dx = \ln|\sec x| + C}$$

# Logarithmic Differentiation

Ex $\frac{d}{dx} \ln \frac{x+1}{\sqrt{x-2}} = \frac{d}{dx}\left[\ln(x+1) - \frac{1}{2}\ln(x-2)\right]$

$= \frac{1}{x+1} - \frac{1}{2(x-2)}$

Ex Differentiate $y = \frac{x^3\sqrt{x^4-5}}{(x+1)(x-3)^7}$

Trick: compute $\frac{d}{dx} \ln y$

$\ln y = \ln\left(\frac{x^3\sqrt{x^4-5}}{(x+1)(x-3)^7}\right) = 3\ln x + \frac{1}{2}\ln(x^4-5) - \ln(x+1) - 7\ln(x-3)$

$\frac{d}{dx}\ln y = \frac{3}{x} + \frac{4x^3}{2(x^4-5)} - \frac{1}{x+1} - \frac{7}{x-3}$

$\|$

$\frac{1}{y} \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = y \cdot \left(\frac{3}{x} + \frac{2x^3}{x^4-5} - \frac{1}{x+1} - \frac{7}{x-3}\right)$

$= \frac{x^3\sqrt{x^4-5}}{(x+1)(x-3)^7}\left(\frac{3}{x} + \frac{2x^3}{x^4-5} - \frac{1}{x+1} - \frac{7}{x-3}\right)$

# The Natural Exponential Function (§5.3)

## Graphing $\ln(x)$:

Observe:

a) $\lim_{x \to \infty} \ln x = \infty$ and b) $\lim_{x \to 0^+} \ln x = -\infty$

To see a): $\ln(2^n) = n \ln(2)$

Now, $\ln(2) > 0$, so $\lim_{n \to \infty} n \ln(2) = \ln(2) \cdot \lim_{n \to \infty} n = \infty$

A similar argument can be made for b.

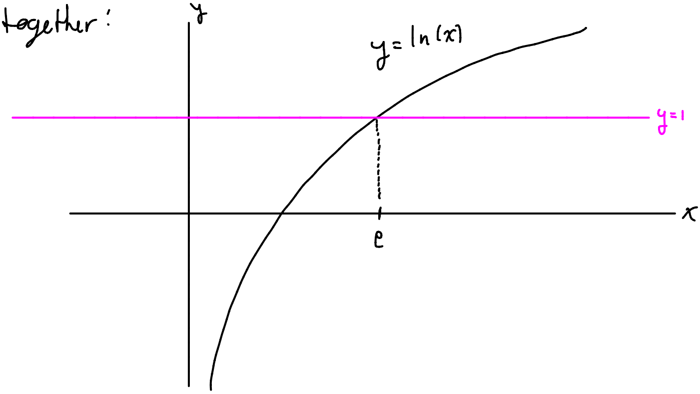
Now, $\frac{d}{dx} \ln x = \frac{1}{x} > 0$ for $x > 0$, so $\ln(x)$ is increasing

we also know that $\frac{d^2}{dx^2} \ln x = \frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2} < 0$

so $\ln x$ is concave down.

Further $\lim_{x \to \infty} \ln x = \infty$ and $\lim_{x \to 0^+} = -\infty$

Putting it all together:




**Def** The number e is defined by the property
$$\ln(e) = 1.$$

$(e = 2.71828\ldots)$

As $\frac{d}{dx} \ln x > 0$ for all $x$, $\ln x$ is increasing

$\Rightarrow$ $\ln x$ has an inverse

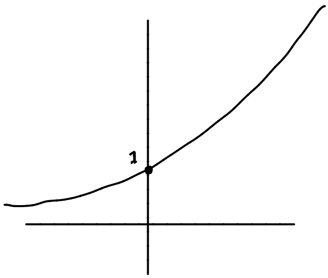
Now observe that $\ln(e^x) = x \ln(e) = x$

This means that the function $e^x$ is the inverse of $\ln x$.

<u>Def</u> $e^x$ is the <u>natural exponential function</u>.

$$\boxed{\begin{array}{ll} \cdot\ e^x = y \iff \ln y = x & \\ \cdot\ e^{\ln x} = x & x > 0 \\ \cdot\ \ln(e^x) = x & \text{for all } x \end{array}}$$

Graph of $y = e^x$




<u>Ex</u> $e^0 = 1$ b/c $\ln(1) = 0$.