Ex. Solve the equation $e^{3x-2} = 9$ for x:

$$e^{3x-2} = 9$$

$$\Rightarrow \ln(e^{3x-2}) = \ln(9)$$

$$\Rightarrow 3x - 2 = \ln(9)$$

$$\Rightarrow x = \frac{\ln(9)+2}{3}$$

Laws of exponents If x and y are real numbers and r is rational, then

- $e^{x+y} = e^x \cdot e^y$
- $e^{x-y} = \frac{e^x}{e^y}$
- $(e^x)^r = e^{r \cdot x}$

These laws can be derived from the laws of logarithms.

Differentiation

Thm $\frac{d}{dx} e^x = e^x$

pf $\ln(e^x) = x$

$$\Rightarrow \frac{d}{dx} \ln(e^x) = \frac{d}{dx} x$$

$$\Rightarrow \frac{1}{e^x} \cdot \frac{d}{dx} e^x = 1$$

$$\Rightarrow \frac{d}{dx} e^x = e^x.$$

Ex $\frac{d}{dx} e^{x^2} = e^{x^2} \cdot \frac{d}{dx} x^2 = 2x e^{x^2}$

Integration

As $\frac{d}{dx} e^x = e^x$, we have

$$\boxed{\int e^x \, dx = e^x + C}$$

Ex $\int x e^{x^2} \, dx = \int \frac{1}{2} e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$

$u = x^2$

$du = 2x \, dx$

Ex $\int e^{-3x} \, dx = -\frac{1}{3} e^{-3x} + C$

Exponential Growth and Decay (§5.5)

An equation involving derivatives is called a differential equation.

An important example, which is the focus of this section is:

$$\boxed{\frac{dy}{dt} = ky}$$

This equation is used to describe quanties that grow or decay at a rate proportional to their size.

Solution: $y = Ce^{kt}$

Check $\frac{dy}{dt} = (Ce^{kt}) \cdot k = k \cdot y$

Notice: $y(0) = C \cdot e^0 = C$, So C is the initial value of the function.

Thm The only solutions of the equation $\frac{dy}{dt} = ky$ are the functions $y(t) = y(0)e^{kt}$.

Population Growth

Let $P(t)$ denote the population of some group.

The rate at which $P(t)$ grows is proportional to its size. (Think about cells dividing.)

So, $\frac{dP}{dt} = kP$

k is called the relative growth rate

Ex. The population of the world in 1950 was 2560 million and 3040 million in 1960.

a) What is the relative growth rate?

b) Estimate the population in 2000 and 2020.

a) $P(t) = Ce^{kt}$, we view 1950 as $t=0$, so $C = P(0) = 2560$ mil.

Now, $3040 = P(10) = 2560e^{k(10)}$

$$\Rightarrow e^{10k} = \frac{3040}{2560}$$

$$\Rightarrow 10k = \ln \frac{3040}{2560}$$

$$\Rightarrow k = \frac{1}{10}\ln\left(\frac{3040}{2560}\right) \approx \boxed{0.017185}$$

This says that the population growth rate is ~1.7%.

b) From a) we have $P(t) = 2560e^{0.017185t}$

At the year 2000, $t = 50$

So $P(50) \approx 6045$ million

(actual population in 2000 was 6145 million)

and for 2020

$P(70) \approx 8524$ million

Radioactive Decay

Let $m(t)$ denote the mass of a radioactive substance.

$$\frac{dm}{dt} = km$$

where $k < 0$ in this case, denoting decay instead of growth.

Def Half-life is the time required for half of any given quantity to decay.

Ex. 239 PU (plutonium) is a nuclear waste by product w/ a half life of 24000 years.

What percentage of nuclear waste created today will be here in 1000 years?

$$\frac{1}{2} = e^{k(2400)} \Rightarrow \ln \tfrac{1}{2} = k(24000)$$

$$\Rightarrow k = \frac{\ln \frac{1}{2}}{24000} \approx -0.000029$$

$$\frac{P(1000)}{P(0)} \approx e^{(-0.000029)(1000)} \approx 0.9715 \quad \text{or} \quad 97.15\%$$