Newton's Law of Cooling

Let $T(t)$ be the temperature above the surrounding temperature of an object.

$$\frac{dT}{dt} = kT$$

Here, the object cools, so $k < 0$.

Ex, If the temperature of the room is $70°F$ and it takes 3 minutes for boiling water to cool to $185°F$, then how long does it take to reach $100°F$?

First, we need to find k.

$$T(0) = 212 - 70 = 142$$

$$\Rightarrow T(t) = 142 e^{kt}$$

Now, $185 - 70 = 142 e^{3k}$

$$\Rightarrow 3k = \ln \frac{115}{142}$$

$$\Rightarrow k = \frac{1}{3} \ln \frac{115}{142} \approx -0.07$$

So, $100 - 70 = 142 e^{-0.07t}$

$$\Rightarrow -0.07t = \ln \frac{30}{142}$$

$$\Rightarrow t \approx 22.2 \text{ minutes}$$

The number e as a limit

Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.

Now, $f'(1) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \cdot \ln(1+h)$$

$$= \lim_{h \to 0} \ln\left[(1+h)^{1/h}\right] = 1$$

So $e = e^1 = e^{\lim\limits_{h \to 0} \ln\left[(1+h)^{1/h}\right]} = \lim\limits_{h \to 0} e^{\ln\left[(1+h)^{1/h}\right]} = \lim\limits_{h \to 0} (1+h)^{1/h}$

$$\boxed{e = \lim_{x \to 0} (1+x)^{1/x}}$$

$$\boxed{e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x}$$

Continuously compounded interest

If \$1000 is invested at 6% interest, compounded annually,

then after 1 year it is worth $1000(1.06) = \$1060$,

after 2 years it is worth $1000(1.06)(1.06) = 1000(1.06)^2 = \$1123.60$,

and after t years it is worth $1000(1.06)^t$

General formula: $A_0(1+r)^t$ (compounded annually)

Most investments are compounded more frequently.

If it is compounded n times a year, then there nt compounding periods in t years.

Value of investment after t years $= A_0\left(1+\frac{r}{n}\right)^{nt}$ (compounded n times per year)

Ex. After 3 years at 6% interest a \$1000 investment is worth:

$\$1000(1.06)^3 = \$1191.02$ w/ annual compounding

$\$1000\left(1+\frac{0.06}{2}\right)^{3\cdot 2} = \$1194.05$ w/ Semiannual "

$\$1000\left(1+\frac{0.06}{12}\right)^{3\cdot 12} = \$1195.62$ w/ monthly "

$\$1000\left(1+\frac{0.06}{365}\right)^{3\cdot 365} = \$1197.20$ w/ daily "

↑ most investments are compounded daily

If we let $n \to \infty$, then we continuous compounding:

$$A(t) = \lim_{n\to\infty} A_0\left(1+\frac{r}{n}\right)^{nt}$$

$$= \lim_{n\to\infty} A_0\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt}$$

$$= A_0\left[\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt}$$

$$= A_0\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{rt} \qquad m=\frac{n}{r}$$

$$= A_0 e^{rt}$$

Back to examplo $\$1000e^{0.06(3)} = \$1197.22$

Observe: $\quad \dfrac{dA}{dt} = \dfrac{d}{dt} A_0e^{rt} = r\left(A_0e^{rt}\right) = rA$

$\Rightarrow$ Investments compounded continuously grow at a rate proportional to their size.

Ex If your credit card has an interest rate of 18% and compounds continuously, how much will you owe if you spend $1000 on the new iPhone XS if you ignore your credit card bill for 5 years?

$$1000e^{0.18\cdot 5} = \$2459.60$$