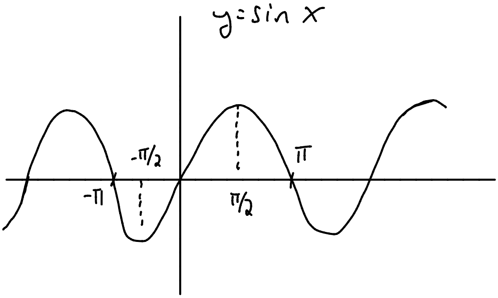
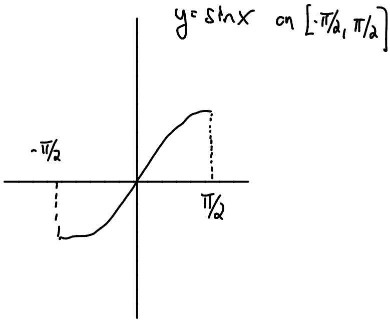
Inverse Trig Functions (§5.6)





$f(x) = \sin x$ is not one-to-one.

However, if we restrict the domain of $\sin x$ to $[-\pi/2, \pi/2]$ then it does become one-to-one

Def $\underline{\arcsin(x)}$ is the inverse of $\sin x$ w/ domain restricted to $[-\pi/2, \pi/2]$

- $\arcsin(x) = y \iff \sin y = x$ and $-\pi/2 \le y \le \pi/2$
- Domain $\arcsin(x)$ is $[-1, 1]$
- Range $\arcsin(x)$ is $[-\pi/2, \pi/2]$

(Note: Another notation for $\arcsin(x)$ is $\sin^{-1}(x)$. You will see this in the textbook. So $\arcsin(x) = \sin^{-1}(x)$

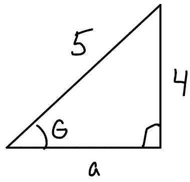
In book, don't confuse $\sin^{-1}(x)$ w/ $\frac{1}{\sin(x)}$)

Ex. $\arcsin(0) = 0$ $\quad \arcsin(-1) = -\pi/2$ $\quad \arcsin(1/2) = \pi/6$

$\arcsin(1) = \pi/2$ $\quad \arcsin\left(\frac{\sqrt{2}}{2}\right) = \pi/4$

Ex Find $\tan(\arcsin(4/5))$

Let $\theta = \arcsin(4/5)$



So, $\sin\theta = 4/5 = \frac{\text{opposite}}{\text{hypotenuse}}$

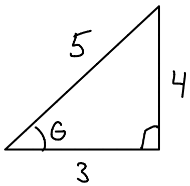
we use this to draw a reference triangle

We then use the Pythagorean Thm to fill in the missing side!

$a^2 + 4^2 = 5^2$

$a^2 = 25 - 16 = 9$

$a = 3$



Now, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$

Ex $\arcsin(\sin(2\pi)) = \arcsin(0) = 0 \neq 2\pi$

$\sin(\arcsin x) = x$ for $-1 \leq x \leq 1$

$\arcsin(\sin x) = x$ for $-\pi/2 \leq x \leq \pi/2$

Differentiation

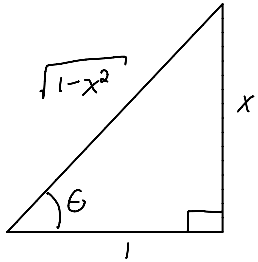
Let $\theta = \arcsin(x)$.

Then $\sin\theta = x$

$$\Rightarrow \frac{d}{dx}\sin\theta = \frac{d}{dx}x$$

$$\Rightarrow \cos\theta \frac{d}{dx}\theta = 1$$

$$\Rightarrow \frac{d}{dx}\theta = \frac{1}{\cos\theta}$$



To find $\cos\theta$, we draw a reference triangle like before using $\sin\theta = \frac{x}{1} = \frac{opp}{hyp}$

Now $\cos\theta = \frac{adj}{hyp} = \frac{1}{\sqrt{1-x^2}}$

Plugging into $\frac{d}{dx}\theta = \frac{1}{\cos\theta}$, we obtain

$$\boxed{\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}}$$

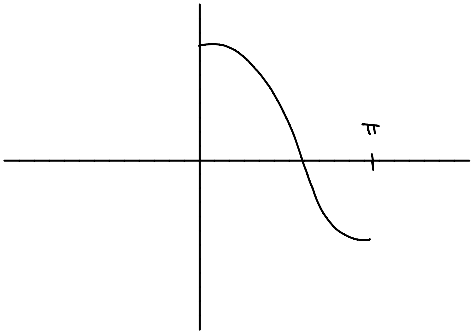
Ex: Let $f(x) = \arcsin(x^2-1)$.

a) Find the domain of f. (A: $[-\sqrt{2}, \sqrt{2}]$)

b) Find $f'(x)$ (A: $\frac{2x}{\sqrt{1-(x^2-1)^2}} = \frac{2x}{\sqrt{2x^2-x^4}} = \frac{2}{\sqrt{2-x^2}}$)

c) Find the domain of f'. (A: $(-\sqrt{2}, \sqrt{2})$)

Now, Cosine:



$\cos(x)$ restricted to the interval $[0, \pi]$ is one-to-one

<u>Def</u> The function <u>arccos(x)</u> is defined to be the inverse of $\cos(x)$ on $[0, \pi]$.

$\arccos(x) = y \iff \cos(y) = x$ and $0 \le y \le \pi$

$\arccos(\cos x) = x$ for $0 \le x \le \pi$

$\cos(\arccos x) = x$ for $-1 \le x \le 1$

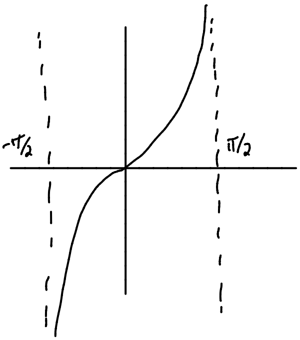
Domain of $\arccos x = [-1, -1]$

Range of $\arccos x = [0, \pi]$

Similar to arcsin, we have

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

Now, tangent:



$\tan(x)$ is one-to-one on the interval $(-\pi/2, \pi/2)$

Def $\underline{\text{arctan}(x)}$ is the inverse of $\tan(x)$ on the interval $(-\pi/2, \pi/2)$

Domain of $\arctan x$ = all real numbers

Range of $\arctan x$ = $(-\pi/2, \pi/2)$

$\lim_{x \to -\infty} \arctan(x) = -\pi/2$, $\lim_{x \to \infty} \arctan(x) = \pi/2$

$$\boxed{\frac{d}{dx} \arctan x = \frac{1}{1+x^2}}$$

Table of inverse trig functions

Function	Domain	Range	Derivative
$\arcsin x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$[-1, 1]$	$[0, \pi]$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$	$\frac{1}{1+x^2}$
$\operatorname{arccsc} x$	$\lvert x \rvert \geq 1$	$(0, \pi/2] \cup (\pi, 3\pi/2]$	$-\frac{1}{x\sqrt{x^2-1}}$
$\operatorname{arcsec} x$	$\lvert x \rvert \geq 1$	$[0, \pi/2) \cup [\pi, 3\pi/2)$	$\frac{1}{x\sqrt{x^2-1}}$
$\operatorname{arccot} x$	$(-\infty, \infty)$	$(0, \pi)$	$-\frac{1}{1+x^2}$