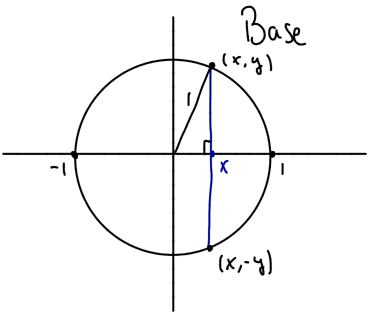
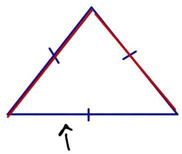
Other cross-sections

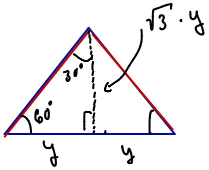
Ex Find the volume of the solid whose base is a circle of radius 1 and w/ parallel cross sections perpendicular to the base are equilateral triangles.



Cross Section at x



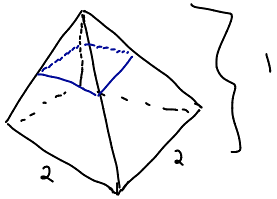
This is the edge we see in the base, So its length is $2y = 2\sqrt{1-x^2}$

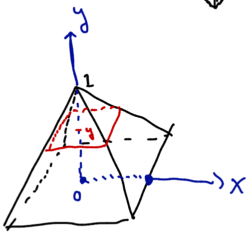


$$A(x) = \frac{1}{2} b \cdot h = \frac{1}{2}(2y) \cdot (\sqrt{3}\, y) = \sqrt{3} \cdot y^2 = \sqrt{3}(1-x^2)$$

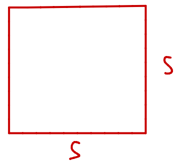
$$\Rightarrow V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}(1-x^2)\,dx = \sqrt{3}\left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^{1} = \frac{4\sqrt{3}}{3}$$

Ex Find the volume of the pyramid w/ square base of side length 2 and whose height is 1.



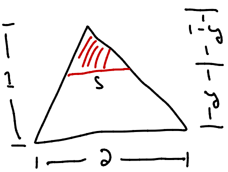


Cross section at y is a square



Need to find s

Let's look at the face of the pyramid



The red triangle is similar to the bigger triangle, so

$$\frac{2}{s} = \frac{1}{1-y}$$

$$\Rightarrow s = 2(1-y)$$

$$\Rightarrow A(y) = s^2 = 4(1-y)^2$$

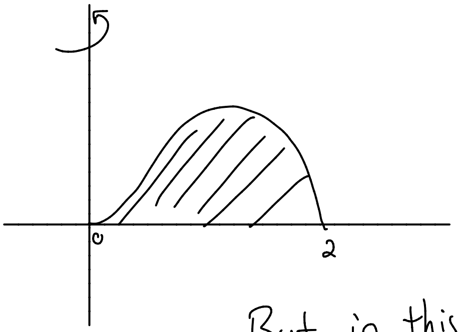
$$\Rightarrow V = \int_0^1 4(1-y)^2\,dy = \int_0^1 4(1-2\quad y^2)\,dy$$

$$= 4\left(y - y^2 + \frac{y^3}{3}\right)\Big|_0^1$$

$$= \frac{4}{3}$$

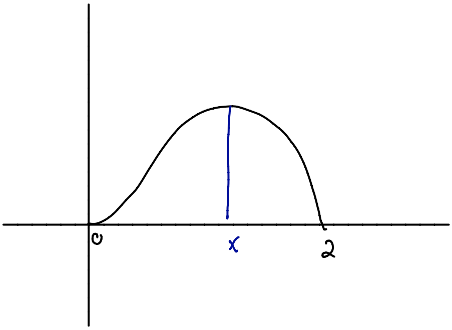
Volumes by Cylindrical Shells (§7.3)

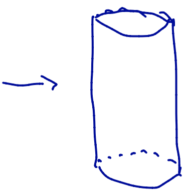
Ex Find the volume of the solid obtained by revolving the region bounded by $y = 2x^2 - x^3$ and $y = 0$ about the y-axis.



If we want to use cross section perpendicular to the y-axis then we need to integrate wrt to y.

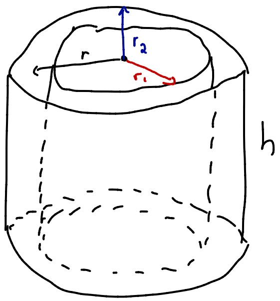
But in this case we need to solve for x in terms of y. It's not clear how to do this, so we need another method.





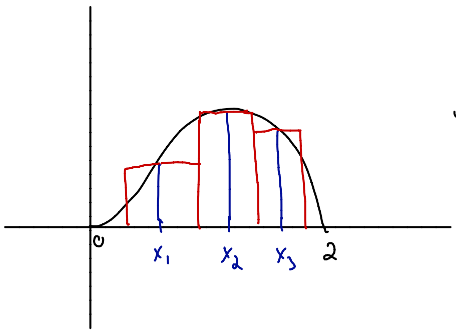
If thicken this can we get a cylindrical shell.

Cylindrical Shell:



$$V = V_2 - V_1 = \pi r_2^2 h - \pi r_1^2 h$$
$$= \pi(r_2^2 - r_1^2)h$$
$$= \pi(r_2 + r_1)(r_2 - r_1)h$$
$$= 2\pi\left(\frac{r_2 + r_1}{2}\right)h(r_2 - r_1)$$
$$\boxed{V = 2\pi r h \Delta r}$$

Now, we can approximate our solid by cylindrical shells.

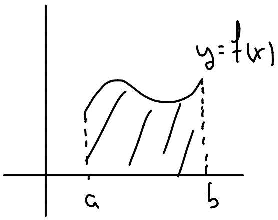


$$\rightsquigarrow V = \lim_{n\to\infty} \sum_{j=0}^{n} 2\pi x_j \, f(x_j) \, \Delta x$$

radius ↑ x_j, height ↑ $f(x_j)$

$$\Rightarrow V = \int_0^2 2\pi x f(x)\, dx$$
$$= \int_0^2 2\pi x(2x^2 - x^3)\, dx$$
$$= 2\pi \int_0^2 (2x^3 - x^4)\, dx = \frac{16}{5}\pi$$

More generally, the volume of the solid obtained by revolving the region bounded by the curves $y=f(x)$, $y=0$, $x=a$, and $x=b$

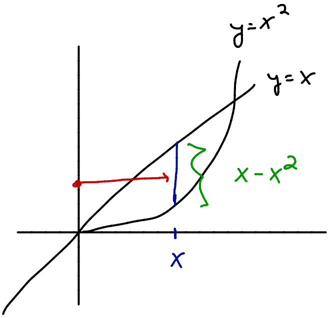


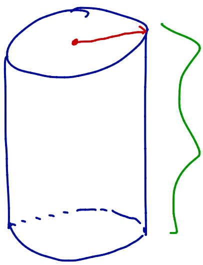
has volume $V=\int_a^b 2\pi x\, f(x)\, dx$

where $0 \leq a < b$

In this formula, x is playing the role of the radius of the shell and $f(x)$ the height of the shell.

Ex Find the volume of the solid obtained by rotating the region bounded by $y=x$ and $y=x^2$ about the y-axis.





The radius is x.

The height is $x-x^2$.

$$V=\int_0^1 2\pi x(x-x^2)\,dx$$

$$=2\pi\int_0^1 (x^2-x^3)\,dx$$