Differential Equations (§7.7)

Def A <u>differential equation</u> is an equation that contains an unknown function and one or more of its derivatives.

Ex
- $y' = ky$ (Exponential growth / decay)
- $y' = xy$
- $y'' + y' + y = 0$

Def The <u>order</u> of a diff eq is the highest order of a derivative in the equation.

Def A function $f(x)$ is called a <u>solution</u> of a diff eq if the equation is satisfied when $y = f(x)$.

Ex Solve the diff eq $\frac{dy}{dx} = x^2$.

Here, by solve, we mean find all solutions.

we know that if $\frac{dy}{dx} = x^2$, then

then general solution is $y = \frac{x^3}{3} + C$.

Ex. Find a solution to $y'' = -y$.

$y = \sin x$ is a solution.

Def A separable equation is a first-order diff eq that can be written as $\frac{dy}{dx} = g(x) f(y)$.

Solving separable equations:

$$\frac{dy}{dx} = g(x) f(y)$$

Rewrite so all the y's on one side and all the x's on the other

$$\frac{dy}{f(y)} = g(x)\,dx$$

Then integrate both sides:

$$\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

Ex. (a) Solve the diff eq $\frac{dy}{dx} = \frac{x^2}{y}$

(b) Find the solution satisfying $y(0) = 2$

(a) $\frac{dy}{dx} = \frac{x^2}{y} \Rightarrow y\,dy = x^2\,dx \Rightarrow \int y\,dy = \int x^2\,dx$

$\Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + C$ ← Technically, both sides get a $+C$, but we combine the constants into a single constant.

$\Rightarrow y^2 = \frac{2}{3}x^3 + 2C$

$\Rightarrow y = \pm\sqrt{\frac{2}{3}x^3 + 2C}$ Here $2C$ is another constant, let's name it K

$\Rightarrow y = \pm\sqrt{\frac{2}{3}x^3 + K}$

(b) When $x=0$, $y=2$. First observe that this means we choose the positive branch of the square root:

$y = \sqrt{\frac{2}{3}x^3 + K}$, Now setting $x=0$ and $y=2$, we have $2 = \sqrt{K} \Rightarrow K = 4$.

$\Rightarrow y = \sqrt{\frac{2}{3}x^3 + 4}$

The type of problem as in part (b) is called an <u>initial-value problem</u> and asks to find a particular solution of a diff eq.

<u>Ex</u> (a) Solve $\frac{dy}{dx} = xy$.

(b) Find the solution satisfying $y(0) = 7$

a) If $y \neq 0$, then we have $\frac{dy}{y} = x\,dx$

$\Rightarrow \int \frac{dy}{y} = \int x\,dx \quad \Rightarrow \quad \ln|y| = \frac{x^2}{2} + C$

$\Rightarrow |y| = e^{x^2/2 + C} = e^C e^{x^2/2}$

$\Rightarrow y = \pm e^C e^{x^2/2}$

Let $A = \pm e^C$, then $y = Ae^{x^2/2}$ w/ $A \neq 0$.

But, $y=0$ is a solution also, so we can conclude

$y = Ae^{x^2/2}$ is the general solution to $\frac{dy}{dx} = xy$

(b) Setting $x=0$ and $y=7$ we have

$$7 = A$$

$$\Rightarrow y = 7e^{x^2/2}$$

Ex. Solve the diff eq $\frac{dy}{dx} = \frac{\sin x}{2+e^y}$

$$\frac{dy}{dx} = \frac{\sin x}{2+e^y}$$

$$\Rightarrow (2+e^y)\,dy = \sin x\,dx$$

$$\Rightarrow \int (2+e^y)\,dy = \int \sin x\,dx$$

$$\Rightarrow 2y + e^y = -\cos x + C$$

We have to stop here as we can't solve for y.

But this still gives y as an implicit function of x.