Sigma Notation (Appendix B)

Sigma is a greek letter: Σ, σ
uppercase, lower case

(irrelevant: Σ makes the same sound as S)

Def If $a_m, a_{m+1}, \dots, a_n$ are real numbers and m and n are integers such that $m \le n$, then

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_{n-1} + a_n$$

here i is the index of summation

Or, w/ function notation, $\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + \dots + f(n-1) + f(n)$

Examples

- $\sum_{i=3}^{6} (i+1)^2 = \underbrace{(3+1)^2}_{i=3} + \underbrace{(4+1)^2}_{i=4} + \underbrace{(5+1)^2}_{i=5} + \underbrace{(6+1)^2}_{i=6}$
  $= 4^2 + 5^2 + 6^2 + 7^2 = 16 + 25 + 36 + 49 = 126$

- $\sum_{j=1}^{10} j = 1+2+3+4+5+6+7+8+9+10 = 55$

- $\sum_{k=2}^{5} \frac{1}{2^k} = \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$
  $= \frac{8+4+2+1}{32} = \frac{15}{32}$

- $\sum_{i=4}^{7} 5 = \underbrace{5}_{i=4} + \underbrace{5}_{i=5} + \underbrace{5}_{i=6} + \underbrace{5}_{i=7} = 20$

Ex, Find another way of writing

$$\sum_{i=3}^{6}(i+1)^2$$

Properties of Summation

1) $\sum_{i=m}^{n} ca_i = c\sum_{i=m}^{n} a_i$, where c is a constant (meaning it does not depend on i)

2) $\sum_{i=m}^{n}(a_i+b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i$

3) $\sum_{i=m}^{n}(a_i-b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i$

4) $\sum_{i=m}^{l} a_i + \sum_{i=l+1}^{n} a_i = \sum_{i=m}^{n} a_i$

Reason

1) $\sum_{i=m}^{n} ca_i = ca_m + ca_{m+1} + \cdots + ca_n = c(a_m + \cdots + a_n) = c\sum_{i=m}^{n} a_i$

2) $$\begin{aligned}\sum_{i=m}^{n}(a_i+b_i) &= (a_m+b_m) + (a_{m+1}+b_{m+1}) + \cdots + (a_n+b_n)\\ &= (a_m+a_{m+1}+\cdots+a_n) + (b_m+\cdots+b_n)\\ &= \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i\end{aligned}$$

Problem Find a formula for $\sum_{i=1}^{n} i$.

$\left(A: \sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right)$

Common Summation Formulas:

- $\sum_{i=1}^{n} 1 = n$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$

Examples

- Evaluate $\sum_{i=1}^{10} 2i(i^2+5)$

  (A: 6600)

- Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{i}{n}\right)^2+1\right]$

First, $\sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{i}{n}\right)^2+1\right] = \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2+1\right]$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$

$$= \frac{2}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$

$$= \frac{2}{n^3}\left(\frac{n(n+1)}{2}\right) + \frac{2}{n}\cdot n$$

$$= \frac{n+1}{n^2} + 2 = \frac{1}{n} + \frac{1}{n^2} + 2$$

So, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left[\left(\frac{i}{n}\right)^2+1\right] = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}+2\right) = 2.$