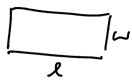
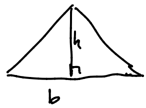
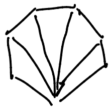
§4.1 Areas and Distances

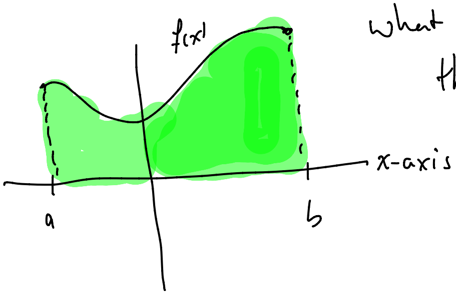
Def The area of a rectangle  is $A = lw$.

From this, you can derive the formula for a triangle: 

$$A = \frac{1}{2}bh.$$

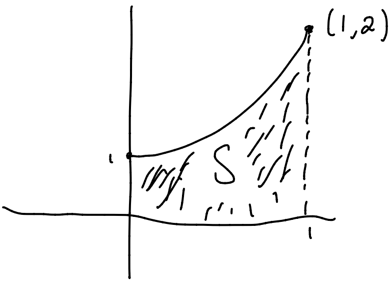
And then for polygons 

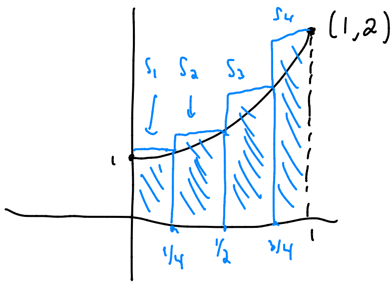
Goal: Given a continuous function $f(x)$ on an interval $[a,b]$, find the area between the x-axis and the graph of $f(x)$



what is the area of the green region?

Ex. Approximate the area under the curve $f(x) = x^2+1$ on $[0,1]$





We approximate the region S by 4 rectangles S_1, S_2, S_3, S_4

Notice that S is contained in the union of the 4 rectangles, so if A is the area of S, then

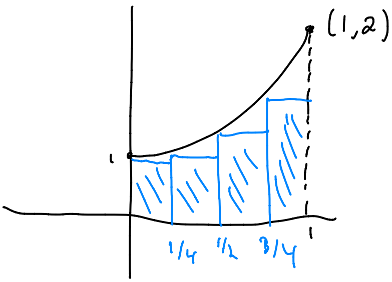
$$A \leq \text{area}(S_1) + \text{area}(S_2) + \text{area}(S_3) + \text{area}(S_4)$$

$$\Rightarrow A \leq \tfrac{1}{4} f(1/4) + \tfrac{1}{4} f(1/2) + \tfrac{1}{4} f(3/4) + \tfrac{1}{4} f(1)$$

$$\Rightarrow A \leq \tfrac{1}{4}(1/16 + 1) + \tfrac{1}{4}(\tfrac{1}{4} + 1) + \tfrac{1}{4}(9/16 + 1) + \tfrac{1}{4}(2)$$

$$\Rightarrow A \leq \frac{47}{32}$$

Notation, we call the sum we just compute R_4:
R for right, and 4 for the number of rectangles.



$$L_4 = \tfrac{1}{4} f(0) + \tfrac{1}{4}(f(1/4)) + \tfrac{1}{4} f(1/2) + \tfrac{1}{4} f(3/4)$$

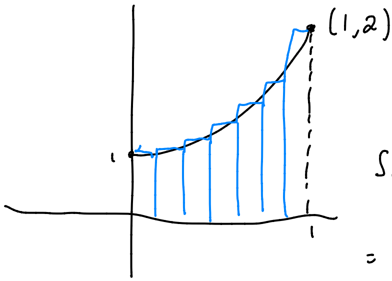
$$= \frac{39}{32}$$

$$L_4 \leq A \leq R_4$$

How do we make our estimate better?

Add more rectangles!

Let's calculate R_n



With n rectangles, the width of each rectangle is $\frac{1}{n}$.

So, $R_n = \frac{1}{n} f(\frac{1}{n}) + \frac{1}{n} f(\frac{2}{n}) + \cdots + \frac{1}{n} f(1)$

$$= \frac{1}{n}\left(\frac{1^2}{n^2}+1\right) + \frac{1}{n}\left(\frac{2^2}{n^2}+1\right) + \cdots \frac{1}{n}\left(\frac{n^2}{n^2}+1\right)$$

$$= \frac{1}{n}\left[\left(\frac{1^2}{n^2}+1\right) + \left(\frac{2^2}{n^2}+1\right) + \cdots + \left(\frac{n^2}{n^2}+1\right)\right]$$

$$= \frac{1}{n}\left[\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{n^2}{n^2}\right) + n\right]$$

$$= \frac{1}{n}\left[\frac{1}{n^2}\left(1^2+2^2+\cdots+n^2\right) + n\right]$$

$$= \frac{1}{n^3}\left(1^2+\cdots+n^2\right) + 1$$

$$= \left(\frac{1}{n^3}\sum_{i=1}^{n} i^2\right) + 1$$

$$= \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} + 1$$

$$= \frac{(n+1)(2n+1)}{6n^2} + 1$$

$$= \frac{2n^2+3n+1}{6n^2} + 1$$

$$= \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2} + 1$$

$$= \frac{4}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Now, $A \le R_n$ for every n, so in fact

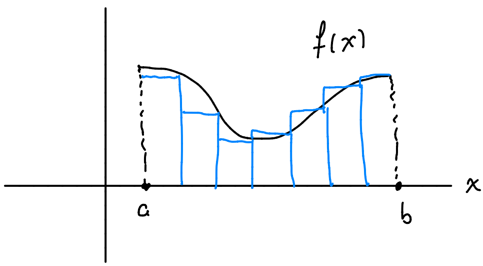
$$A \le \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{4}{3} + \frac{1}{2n} + \frac{1}{6n^2} = \frac{4}{3}.$$

Also, $L_n \le A$ for every n, so

$$\underbrace{\lim_{n\to\infty} L_n}_{=\,4/3} \le A$$

This means $4/3 \le A \le 4/3$ and so, $A = 4/3$!!!

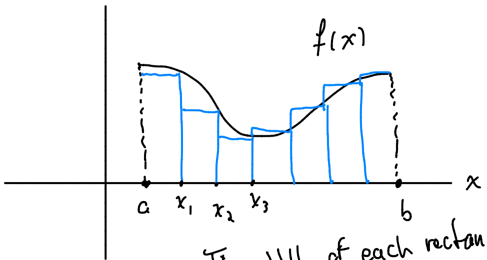
We can do the same thing for a general function.



Let n be an integer.

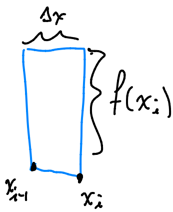
$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

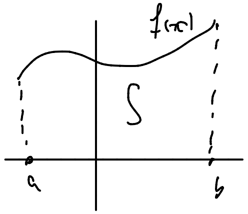


The width of each rectangle is Δx

The i^{th} rectangle has width Δx and height $f(x_i)$



$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$
$$= \sum_{i=1}^{n} f(x_i)\Delta x$$



Def The <u>area</u> A of the region S that lies under the graph of a nonnegative, continuous function f is

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Note: $L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

In fact, you can do this process w/ any increasing number of rectangles and obtain the same limit.