The Definite Integral (§4.2)

Def If f is a function defined on $[a,b]$, the <u>definite integral</u> of f from a to b is the limit

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. If the limit exists then we say f is <u>integrable</u> on $[a,b]$.

The sum $\sum_{i=1}^{n} f(x_i)\,\Delta x$ is an example of a <u>Riemann sum</u>.

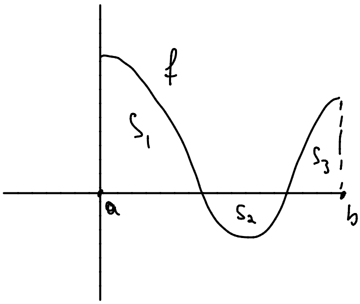
Notation:
- $f(x)$ is called the <u>integrand</u>
- a and b are called the <u>limits of integration</u>
- <u>integration</u> is the process of computing an integral

Theorem If f is continuous on $[a,b]$, or if f has a finite number of jump discontinuities, then f is integrable on $[a,b]$.

Observe, if f is nonnegative on $[a,b]$, then

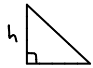
$$\int_a^b f(x)\,dx = \text{area under the graph of } f.$$

In general, $\int_a^b f(x)\,dx$ computes the <u>net area</u> between the graph of f and the x-axis. The net area is the difference of the area above the x-axis and below.

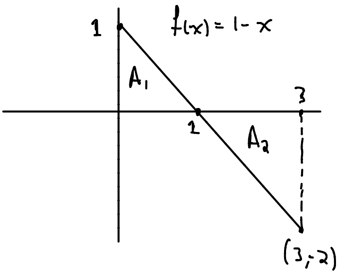


$$\int_a^b f(x)\,dx = \text{Area}(S_1) - \text{Area}(S_2) + \text{Area}(S_3)$$

Recall:

Area() $= \frac{1}{2}bh$

<u>Ex.</u> Compute $\int_0^3 (1-x)\,dx$

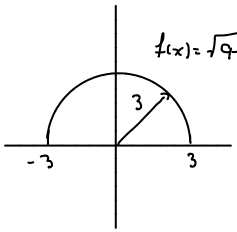


$$A_1 = \frac{1}{2}(1)\cdot(1) = \frac{1}{2}$$

$$A_2 = \frac{1}{2}(2)(2) = 2$$

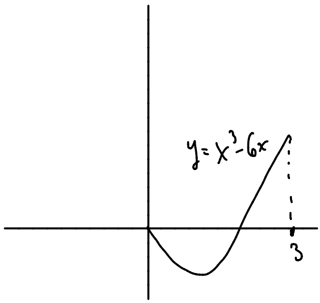
$$\int_0^3 (1-x)\,dx = A_1 - A_2 = \frac{1}{2} - 2 = -\frac{3}{2}$$

Ex Compute $\int_{-3}^{3} \sqrt{9-x^2}\, dx$



$$\int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{1}{2}\pi(3)^2 = \frac{9\pi}{2}$$

Ex Compute $\int_{0}^{3} (x^3 - 6x)\, dx$



$$\int_0^3 (x^3 - 6x)\, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right) \frac{3}{n}$$

$$= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

$$= \lim_{n \to \infty} \frac{3}{n} \sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18}{n} i\right]$$

$$= \lim_{n \to \infty} \left[\frac{81}{n^4} \sum_{i=1}^{n} i^3 - \frac{54}{n^2} \sum_{i=1}^{n} i\right]$$

$$= \lim_{n \to \infty} \left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n \to \infty} \left[\frac{81}{4}\left(\frac{n^2+n}{n^2}\right)^2 - 27\left(\frac{n^2+n}{n^2}\right)\right]$$

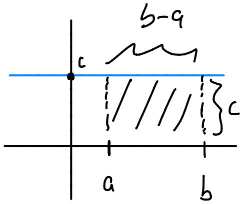
$$= \lim_{n \to \infty} \left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] = \frac{81}{4} - 27 = -\frac{27}{4}$$

Properties of the definite integral

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (Δx changes signs)

- $\int_a^a f(x)\,dx = 0$ ($\Delta x = 0$)

- $\int_a^b c\,dx = c(b-a)$ where c is any constant

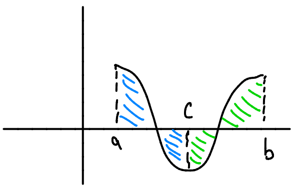


- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

- $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ where c is any constant

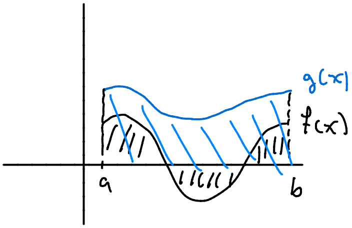
- $\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

- If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b (g(x) - f(x)) \geq 0$



- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$