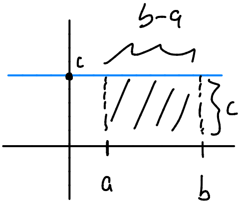
Properties of the definite integral

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (Δx changes signs)

- $\int_a^a f(x)\,dx = 0$ ($\Delta x = 0$)

- $\int_a^b c\,dx = c(b-a)$ where c is any constant

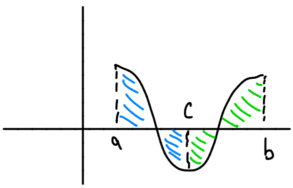


- $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

- $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ where c is any constant

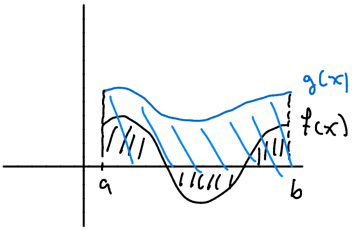
- $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$



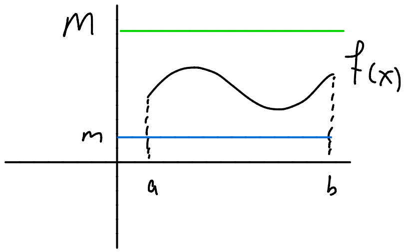
- If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

- If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\int_a^b (g(x) - f(x)) \geq 0$



- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$



Evaluating Definite Integrals (§4.3)

Evaluation Theorem: If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is any antiderivative of f (that is, $F' = f$).

Proof Let n be a positive integer.

Divide $[a,b]$ into n subintervals w/ endpoints

$x_0(=a), x_1, x_2, \ldots, x_n(=b)$ and w/ equal length $\Delta x = \frac{b-a}{n}$

$$\begin{aligned} F(b)-F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) + \cdots + F(x_2) - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} \left[F(x_i) - F(x_{i-1}) \right] \end{aligned}$$

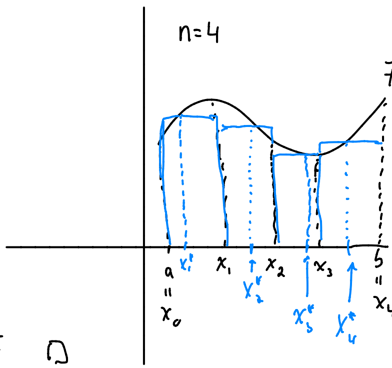
Now, F is continuous on $[a,b]$ (b/c it's differentiable) and so we apply the Mean Value Theorem to each interval of the form $[x_{i-1}, x_i]$ to obtain x_i^* w/

$$\begin{aligned} F(x_i) - F(x_{i-1}) &= F'(x_i^*)(x_i - x_{i-1}) \\ &= f(x_i^*)\,\Delta x \end{aligned}$$

So, $F(b) - F(a) = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

This works for every n, so

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \int_a^b f(x)\,dx \quad \square$$



Notation

$F(x)\Big|_a^b = F(b)-F(a)$

Ex. $\int_0^3 (x^3-6x)\,dx$

Find an antiderivative of $f(x) = x^3-6x$.

$F(x) = \frac{x^4}{4} - 3x^2$ works.

So, $\int_0^3 (x^3-6x)\,dx = F(3)-F(0) = \left(\frac{x^4}{4}-3x^2\right)\Big|_0^3$

$= \left(\frac{3^4}{4} - 3(3)^2\right) - \left(\frac{0^4}{4} - 3(0)^2\right)$

$= \frac{81}{4} - 27 = -\frac{27}{4}$

Ex. $\int_{-1}^2 (9x^2-2x+7)\,dx$

Find an antiderivative of $f(x) = 9x^2-2x+7$

$F(x) = 3x^3 - x^2+7x$ works

So, by the evaluation theorem

$\int_{-1}^2 (9x^2-2x+7)\,dx = F(2)-F(-1)$

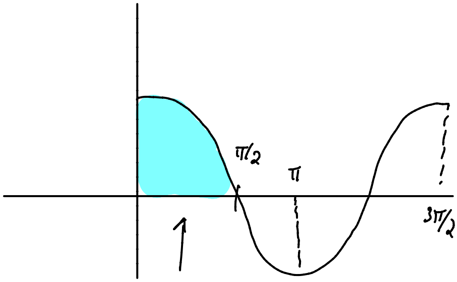
$= \left(3(2)^3-(2)^2+7(2)\right) - \left(3(-1)^3-(-1)^2+7(-1)\right)$

$= 24-4+14-(-3-1-7)$

$= 34+11$

$= 45$

Ex $\displaystyle\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\pi/2} = 1.$



$1 = \displaystyle\int_0^{\pi/2} \cos x \, dx$ is the area of the blue region