Indefinite Integrals

Now that we see a connection between definite integrals and antiderivatives, we denote the entire family of antiderivatives of a function $f(x)$ by $\int f(x)\,dx$, which we call the <u>indefinite integral</u>.

If $F(x)$ is an antiderivative of $f(x)$, then, by definition,

$$\int f(x)\,dx = F(x) + C.$$

For example, $\int x\,dx = \frac{x^2}{2} + C$ b/c $\frac{d}{dx}\left(\frac{x^2}{2} + C\right) = x$.

Table of indefinite integrals

- $\int c f(x)\,dx = c\int f(x)\,dx$, any constant c
- $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int K\,dx = Kx + C$, any Constant K
- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
- $\int \sin x\,dx = -\cos x + C$
- $\int \cos x\,dx = \sin x + C$
- $\int \sec^2 x\,dx = \tan x + C$
- $\int \csc^2 x\,dx = -\cot x + C$
- $\int \sec x \tan x\,dx = \sec x + C$
- $\int \csc x \cot x\,dx = -\csc x + C$

Ex $\int (x^4 + \sin x)\,dx = \frac{x^5}{5} - \cos x + C$

Ex $\int_0^{\pi} (x^4 + \sin x)\,dx = \left(\frac{x^5}{5} - \cos x\right)\Big|_0^{\pi}$

$= \left(\frac{\pi^5}{5} - \cos(\pi)\right) - \left(0 - \cos(0)\right)$

$= \frac{\pi^5}{5} + 1 + 1 = \frac{\pi^5}{5} + 2$

Net Change Thm

$$\int_a^b F'(x)\,dx = F(b)-F(a)$$

Ex. A particle moves along a line w/ velocity $v(t)=t^2-t-6$. (meters/second)

(a) Find the displacement from $t=1$ to $t=4$.

$v(t) = s'(t)$

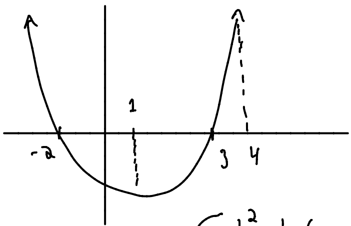
s ↑ displacement

So, $\displaystyle\int_1^4 v(t)\,dt = \int_1^4 (t^2-t-6)\,dt = -\frac{9}{2}$ m

(b) Find the distance traveled from $t=1$ to $t=4$.

Need $\displaystyle\int_1^4 |v(t)|\,dt$.

$t^2-t-6=(t-3)(t+2)$



$$|t^2-t-6| = \begin{cases} t^2-t-6 & t\le -2 \\ -t^2+t+6 & -2\le t\le 3 \\ t^2-t-6 & t\ge 3 \end{cases}$$

$$\int_1^4 |v(t)|\,dt = \int_1^4 |t^2-t-6|\,dt$$

$$= \int_1^3 (-t^2+t+6)\,dt + \int_3^4 (t^2-t-6)\,dt$$

$$= 61/6 \text{ m}$$

The Fundamental Theorem of Calculus

Question: Does every continuous function have an antiderivative?

Consider the function $g(x) = \int_0^x t\,dt$.

$$g(1) = \int_0^1 t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$g(2) = \int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = 2$$

$$g(x) = \int_0^x t\,dt = \frac{t^2}{2}\Big|_0^x = \frac{x^2}{2}$$

Observe: $g'(x) = \frac{d}{dx}\left(\frac{x^2}{2}\right) = x$

Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a,b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is an antiderivative of f, that is, $g'(x) = f(x)$ for $a < x < b$.

(As a consequence, the answer to the above question is yes!)

__Proof__ Recall definition of the derivative:

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Now, $g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$

$$= \left(\int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \right) - \int_a^x f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt$$

So, $\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\,dt$.

We will assume $h > 0$ (the case $h < 0$ is nearly the same).

As f is continuous on $[x, x+h]$, the Extreme Value Theorem says there are numbers u and v so that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and absolute maximum values, respectively, of f on $[x, x+h]$.

So, $mh \leq \int_x^{x+h} f(t)\,dt \leq Mh$

$$\Rightarrow f(u) \cdot h \leq \int_x^{x+h} f(t)\,dt \leq f(v) \cdot h$$

$$\Rightarrow f(u) \leq \frac{1}{h} \int_x^{x+h} f(t)\,dt \leq f(v)$$

$$\Rightarrow f(u) \leq \frac{g(x+h) - g(x)}{h} \leq f(v)$$

Since $x \le u, v \le x+h$, as $h \to 0$ $u \to x$ and $v \to x$.

Therefore,

$$\lim_{h \to 0} f(u) \le \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \le \lim_{h \to 0} f(v)$$

$$\Rightarrow \quad f(x) \le \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \le f(x)$$

$$\Rightarrow \quad g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x) \quad \square$$

<u>Ex</u> $g(x) = \int_7^x e^{t^2} dt$

$g'(x) = e^{x^2}$

What if $g(x) = \int_7^{x^4} e^{t^2} dt$?

In this case, Let $u = x^4$, so

$$g(u) = \int_7^u e^{t^2} dt$$

$$\frac{d}{dx} g(u) = \frac{d}{du}\left(\int_7^u e^{t^2}\right) \cdot \frac{du}{dx} \quad \leftarrow \text{Chain Rule}$$

$$= e^{u^2} \cdot \frac{du}{dx}$$

$$= e^{(x^4)^2} \cdot 4x^3$$

$$= 4x^3 e^{x^8}$$