The Substitution Rule (§4.5)

Recall the chain rule:

If $u = g(x)$ is differentiable, then

$$\frac{d}{dx} f(u) = f'(u) \cdot \frac{du}{dx}$$

The substitution Rule:

If $u = g(x)$ is a differentiable function, then

$$\int f(u) \cdot \frac{du}{dx}\, dx = \int f(u) \cdot du$$

Ex. Find $\int x^2 \sin(x^3)\, dx$

$f(x) = \sin(x)$

$u = x^3$

$\frac{du}{dx} = 3x^2$

$$\int x^2 \sin(x^3) dx = \frac{1}{3} \int 3x^2 \sin(x^3)\, dx$$

$$= \frac{1}{3} \int \underbrace{\sin(x^3)}_{f(u)} \cdot \underbrace{3x^2}_{\frac{du}{dx}} \cdot dx$$

$$= \frac{1}{3} \int \sin(u)\, du$$

$$= -\frac{1}{3} \cos(u) + C$$

$$= -\frac{1}{3} \cos(x^3) + C$$

Check:

$$\frac{d}{dx}\left(-\frac{1}{3} \cos(x^3)\right)$$

$$= \frac{1}{3} \sin(x^3) \cdot 3x^2$$

$$= x^2 \sin(x^3) \checkmark$$

Substitution Rule let's you treat $\frac{du}{dx}$ as a fraction

Again w/ convenient notation:

$$\int x^2 \sin(x^3)\, dx$$

$u = x^3$

$du = 3x^2\, dx \Rightarrow dx = \frac{du}{3x^2}$

$$\int x^2 \sin(x^3)\, dx = \int x^2 \sin(u) \cdot \frac{du}{3x^2}$$

$$= \frac{1}{3}\int \sin(u)\, du$$

$$= -\frac{1}{3}\cos(u) + C$$

$$= -\frac{1}{3}\cos(x^3) + C$$

Example

$$\int x\sqrt{x^2-1}\, dx = \int x\sqrt{u}\, \frac{du}{2x}$$

$u = x^2 - 1$

$du = 2x\, dx$

$$= \frac{1}{2}\int \sqrt{u}\, du$$

$$= \frac{1}{2}\cdot\frac{2}{3}\cdot u^{3/2} + C$$

$$= \frac{1}{3}(x^2-1)^{3/2} + C$$

Ex

$$\int \sqrt{5x+2}\, dx = \int \sqrt{u}\, \frac{du}{5}$$

$u = 5x+2$

$du = 5\, dx$

$$= \frac{1}{5}\int u^{1/2}\, du$$

$$= \frac{1}{5}\cdot\frac{2}{3}\cdot u^{3/2} + C$$

$$= \frac{2}{15}(5x+2)^{3/2} + C$$