Exam 1 Review Solutions

1) Let $f(x) = x^2 - 1$

a) Compute R_5 and L_5 for $f(x)$ on the interval $[-2,-1]$.

$$R_5 = \sum_{j=1}^{5} f(x_i)\cdot \Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{-1-(-2)}{5} = \frac{1}{5}$$

$$x_j = a + j\Delta x = -2 + j\frac{1}{5}$$

$$\Rightarrow R_5 = \frac{1}{5}\left(f\left(-\frac{9}{5}\right) + f\left(-\frac{8}{5}\right) + f\left(-\frac{7}{5}\right) + f\left(-\frac{6}{5}\right) + f(-1) \right)$$

$$= \frac{1}{5}\left[\left(\frac{81}{25}-1\right) + \left(\frac{64}{25}-1\right) + \left(\frac{49}{25}-1\right) + \left(\frac{36}{25}-1\right) + 0\right]$$

Use calculator to finish.

$$L_5 = \frac{1}{5}\left[(4-1) + \left(\frac{81}{25}-1\right) + \left(\frac{64}{25}-1\right) + \left(\frac{49}{25}-1\right) + \left(\frac{36}{25}-1\right)\right]$$

b) Find R_n for $f(x)$ on the interval $[-2,-1]$.

$\Delta x = \frac{1}{n}$ $\quad x_j = -2 + \frac{j}{n}$

$$R_n = \sum_{j=1}^{n} f(x_i)\cdot \Delta x = \frac{1}{n}\sum_{j=1}^{n}\left[\left(-2+\frac{j}{n}\right)^2 - 1\right] = \frac{1}{n}\sum_{j=1}^{n}\left(3 - \frac{4j}{n} + \frac{j^2}{n}\right)$$

$$= \frac{3}{n}\sum_{j=1}^{n} 1 - \frac{4}{n^2}\sum_{j=1}^{n} j + \frac{1}{n^3}\sum_{j=1}^{n} j^2$$

$$= 3 - \frac{4}{n^2}\left(\frac{n^2}{2} + \frac{n}{2}\right) + \frac{1}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

c) Use the limit definition of the definite integral to compute $\int_{-2}^{-1}(x^2-1)\,dx$.

$$\int_{-2}^{-1}(x^2-1)\,dx = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty}\left[3 - \frac{4}{n^2}\left(\frac{n^2}{2}+\frac{n}{2}\right) + \frac{1}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\right]$$

$$= \lim_{n\to\infty}\left[3 - 2 + \frac{2}{n} + \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right]$$

$$= 3 - 2 + \frac{1}{3}$$

$$= \frac{4}{3}$$

2) Calculate the following definite integrals:

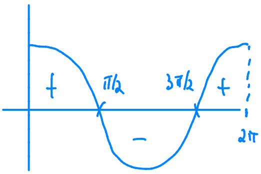
a) $\int_{-2}^{-1}(x^2-1)\,dx$ b) $\int_{0}^{\pi/2}\sin(x)\,dx$ c) $\int_{0}^{2\pi}|\cos x|\,dx$

We use the evaluation theorem:

a) $$\int_{-2}^{-1}(x^2-1)\,dx = \frac{x^3}{3} - x\Big|_{-2}^{-1} = \left(\frac{(-1)^3}{3} - (-1)\right) - \left(\frac{(-2)^3}{3} - (-2)\right)$$
$$= (1 - 1/3) - \left(2 - \frac{8}{3}\right) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

b) $$\int_{0}^{\pi/2}\sin x\,dx = -\cos x\Big|_{0}^{\pi/2} = -\cos(\pi/2) - (-\cos(0)) = 1.$$

c)



$$\int_0^{2\pi} |\cos x|\,dx = \int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{3\pi/2} \cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$$

$$= 4$$

3) Calculate $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{1+\frac{i}{N}}\right)^2$ (Hint: Use FTC)

$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{1+\frac{i}{N}}\right)^2$ is R_N for $f(x) = \frac{1}{x^2}$ on the interval $[1, 2]$

You can tell $a=1$ and $\Delta x = \frac{1}{N} = \frac{b-a}{N}$, so $b=2$.

So, $\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{1+\frac{i}{N}}\right)^2 = \int_1^2 \frac{1}{x^2}\,dx$

$$= -\frac{1}{x}\Big|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

4) A particle travels in a line w/ velocity $v(t) = t^2 - 4t + 3$.

a) What is the displacement of the particle from $t=0$ to $t=4$?

$$s(4) - s(0) = \int_0^4 (t^2 - 4t + 3) = \frac{t^3}{3} - 2t^2 + 3t\Big|_0^4 = \frac{64}{3} - 32 + 12$$

$$= \frac{4}{3}$$

b) What distance does it travel from t=0 ~~to~~ t=0 to t=4?

$$\text{Distance} = \int_0^4 |t^2-4t+3|\,dt$$

$$t^2-4t+3 = (t-1)(t-3)$$

$$|t^2-4t+3| = \begin{cases} t^2-4t+3 & t \le 1 \\ -t^2+4t-3 & 1 \le t \le 3 \\ t^2-4t+3 & t \ge 3 \end{cases}$$

$$\int_0^4 |t^2-4t+3|\,dt = \int_0^1 (t^2-4t+3)\,dt + \int_1^3 (-t^2+4t-3)\,dt + \int_3^4 (t^2-4t+3)\,dt$$

Finish w/ evaluation theorem

5) a) Find the antiderivative $F(x)$ of $f(x) = \sqrt{x}$ such that $F(1)=1$.

$$F(x) = \frac{2}{3}x^{3/2} + C$$

$$1 = F(1) = \frac{2}{3} + C \Rightarrow C = \frac{1}{3}$$

$$F(x) = \frac{2}{3}x^{3/2} + \frac{1}{3}$$

b) Find $\frac{d}{dx}\int_7^{x^2} \sin(t)\,dt$

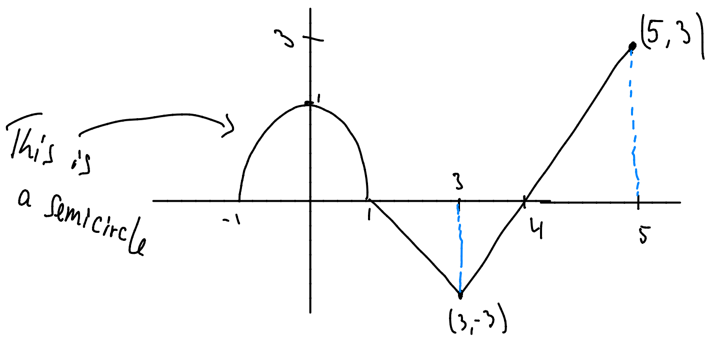
$$\frac{d}{dx}\int_7^{x^2} \sin(t)\,dt = \sin(x^2)\cdot 2x = 2x\sin(x^2)$$

(Used FTC Part I and the Chain rule)

c) Give a formula for the antiderivative $F(x)$ of $f(x) = e^{x^2}$ satisfying $F(2)=0$.

$$F(x) = \int_2^x e^{t^2}\,dt$$

6) The following is the graph of $f(x)$:



Define $A(x) = \int_3^x f(t)\,dt$ and $B(x) = \int_0^x f(t)\,dt$

a) Find $B(-1)$, $B(0)$, $B(1)$, $B(3)$, $B(5)$

$B(-1) = -\pi/4$, $B(0) = 0$, $B(1) = \pi/4$, $B(3) = \pi/4 - 3$, $B(5) = \pi/4 - 3$

b) Find $A(-1)$, $A(0)$, $A(1)$, $A(3)$, $A(5)$

$A(-1) = 3 - \pi/2$, $A(0) = 3 - \pi/4$, $A(1) = 3$, $A(3) = 0$, $A(5) = 0$

c) Find the absolute maximum of $A(x)$ on $[-1, 5]$

The critical numbers of $A(x)$ are $x = 1$ and $x = 4$, so the abs. max occurs at one of $x = -1, 1, 4, 5$.

$A(4) = -3/2$, so we see the abs. max occurs at $x = 1$,

so the absolute max is $A(1) = 3$.

d) Find the equation for $A(x)$ on the interval $[3, 5]$

On $[3,5]$, $f(t) = 3t - 12$ so $A(x) = \int_3^x (3t - 12)\,dt = \left(\frac{3}{2}t^2 - 12t\right)\Big|_3^x$

$$= \frac{3}{2}x^2 - 12x - \left(\frac{27}{2} - 36\right)$$

$$= 3/2\,x^2 - 12x + \frac{45}{2}$$