Sequences (§ 8.1)

Def A sequence is a list of numbers indexed by the natural numbers.

Notation We usually write the numbers in a sequence as $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$

where a_1 is the first term, a_2 the second term, a_n the n^{th} term, etc.

A sequence by definition is infinite, so every term has a next term: a_n is followed by a_{n+1}.

The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

Examples It is often the case that a sequence is defined by giving a formula for the n^{th} term:

- $\{n\}_{n=1}^{\infty}$, $a_n = n$, $\{1, 2, 3, \dots\}$

- $\{5\}_{n=1}^{\infty}$, $a_n = 5$, $\{5, 5, 5, \dots\}$ (constant sequence)

- $\left\{\frac{n^2-1}{n}\right\}_{n=1}^{\infty}$, $a_n = \frac{n^2-1}{n}$, $\{0, \frac{3}{2}, \frac{8}{3}, \frac{15}{4}, \dots\}$

- $\{\cos(n\pi)\}_{n=1}^{\infty}$, $a_n = \cos(n\pi)$, $\{-1, 1, -1, 1, \dots\}$ The same

- $\{(-1)^n\}_{n=1}^{\infty}$, $a_n = (-1)^n$, $\{-1, 1, -1, 1, \dots\}$ Sequence

Example Find a formula for the general term a_n of the sequence $\left\{\frac{4}{3}, -\frac{6}{9}, \frac{8}{27}, -\frac{10}{81}, \dots\right\}$.

$$\left.\begin{aligned}
a_1 &= \frac{4}{3} = \frac{4}{3^1} = (-1)^2 \frac{4}{3^1} = (-1)^2 \frac{2(1)+2}{3^1} \\
a_2 &= -\frac{6}{9} = -\frac{6}{3^2} = (-1)^3 \frac{6}{3^2} = (-1)^3 \frac{2(2)+2}{3^2} \\
a_3 &= \frac{8}{27} = \frac{8}{3^3} = (-1)^4 \frac{8}{3^3} = (-1)^4 \frac{2(3)+2}{3^3} \\
a_4 &= -\frac{10}{81} \underset{(1)}{=} \frac{-10}{3^4} \underset{(2)}{=} (-1)^5 \frac{10}{3^4} \underset{(3)}{=} (-1)^5 \frac{2(4)+2}{3^4}
\end{aligned}\right\} \quad a_n = (-1)^{n+1}\left(\frac{2n+2}{3^n}\right)$$

(1) The denominators are powers of 3.

(2) The sign alternates, a_n is positive when n is odd and negative when n is even $\Rightarrow (-1)^{n+1}$

(3) Numerator increase by 2 $\Rightarrow 2n$ but starts at 2 $\Rightarrow 2n+2$

Not all sequences have formulas:

- $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, \ldots\}$

  $a_n = n^{th}$ prime number

- $\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \ldots\}$

  $a_n = n^{th}$ digit of π.

- $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$ = Fibonacci Sequence

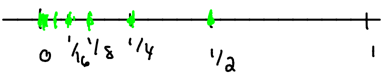
  $a_1 = 1, \; a_2 = 1, \; \underbrace{a_n = a_{n-2} + a_{n-1}}_{\text{Recursive formula}}$

  $a_n = n^{th}$ Fibonacci number

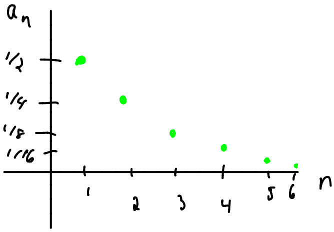
Visualizing Sequences

Ex $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$

1-D



2-D



Observe that as n get large a_n gets closer to 0.

We write $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

Def A sequence $\{a_n\}$ has a <u>limit</u> L and we write

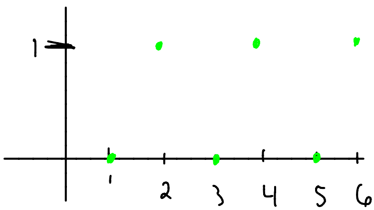
$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n\to\infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence <u>converges</u> (or is <u>convergent</u>). Otherwise, we say the sequence <u>diverges</u> (or is <u>divergent</u>).

<u>Example</u>

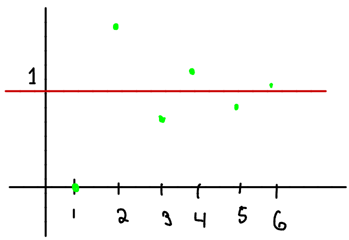
- $\{(-1)^n + 1\}$



Divergent

- $\{(-1)^n \cdot \frac{1}{n} + 1\}$

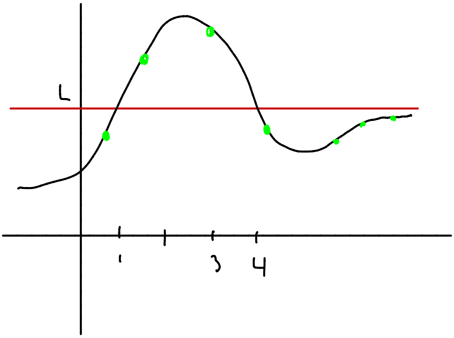
$= \{0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}, \ldots\}$



Convergent $\lim\limits_{n\to\infty} a_n = 1$.

Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, when n is an integer, then

$$\lim_{n\to\infty} a_n = L.$$



Ex, Does $\left\{ \frac{\ln n}{n} \right\}$ converge?

Let $f(x) = \frac{\ln x}{x}$, then $\frac{\ln n}{n} = f(n)$.

So, $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$

L'Hôpital's rule

So yes $\left\{ \frac{\ln n}{n} \right\}$ is convergent.