Definition $\lim_{n\to\infty} a_n = \infty$ means that for every $M>0$ there is $N>0$ such that

$$a_n > M \text{ whenever } n > N.$$

If $\lim_{n\to\infty} a_n = \infty$, we say the sequence $\{a_n\}$ diverges to ∞.

Ex, $\{n^2+1\}_{n=1}^{\infty}$ diverges to ∞!

Given M, take $N = \sqrt{M-1}$.

$\Rightarrow$ If $n > N$, then $a_n = n^2+1 > N^2+1 = M$.

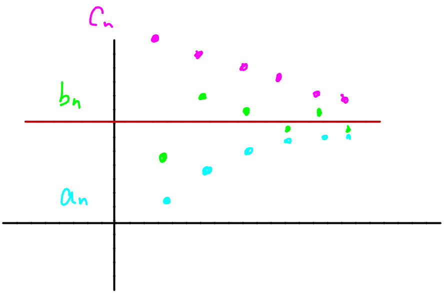
Recalling properties of limits in the context of sequences:

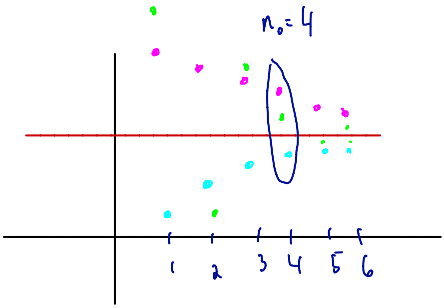
If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is any constant, then:

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} c\, a_n = c \cdot \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
- $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p>0$ and $a_n>0$

Squeeze Thm If $a_n \le b_n \le c_n$ for $n \ge n_0$ and

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = L, \text{ then } \lim_{n\to\infty} b_n = L.$$





Useful consequence of squeeze theorem:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Reason: $-|a_n| \le a_n \le |a_n|$

$$\Rightarrow \underbrace{\lim_{n\to\infty} -|a_n|}_{=0} \le \lim_{n\to\infty} a_n \le \underbrace{\lim_{n\to\infty} |a_n|}_{=0}$$

So, $\lim_{n\to\infty} a_n = 0$.

Ex $$\lim_{n\to\infty}\left(1 + \frac{(-1)^n}{n}\right) = 1 + \lim_{n\to\infty}\frac{(-1)^n}{n} = 1$$

Now, $\left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$

As $\lim_{n\to\infty} \frac{1}{n} = 0$ we know $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

Def · A <u>sequence</u> $\{a_n\}$ is called <u>increasing</u> if $a_n < a_{n+1}$ for all $n \geq 1$.

· " " " " " <u>decreasing</u> " $a_n > a_{n+1}$ " " ".

· A <u>sequence</u> is <u>monotonic</u> if it is either increasing or decreasing.

Ex $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ is decreasing

$$a_{n+1} = \frac{1}{2^{n+1}} = \frac{1}{2\cdot 2^n} = \frac{1}{2}\frac{1}{2^n} = \frac{1}{2}a_n < a_n$$

Ex Is $\left\{\frac{n}{n^2+1}\right\}$ decreasing?

Check: $$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

$$\Updownarrow$$

$$(n+1)(n^2+1) < n\left[(n+1)^2+1\right]$$

$$\Updownarrow$$

$$n^3+n^2+n+1 < n\left[n^2+2n+1+1\right] = n^3+2n^2+2n$$

$$\Updownarrow$$

$$1 < n^2+n$$

This is true for $n \geq 1$, so $\left\{\frac{n}{n^2+1}\right\}$ is decreasing

Alternatively, $f(x) = \frac{x}{x^2+1}$ is decreasing for $x \geq 1$ b/c

$$f'(x) = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \text{for } x > 1$$

Def A sequence $\{a_n\}$ is bounded above if there is a number M such that $a_n \le M$ for every n.

It is bounded below if there is a number m such that

$a_n \ge m$ for every n.

If it bounded from above and below, then $\{a_n\}$ is a bounded sequence.

Monotonic sequence theorem

Every bounded, monotonic sequence is convergent.