Ex Write the number $3.5\overline{23} = 3.5232323\ldots$ as a fraction.

$$3.5\overline{23} = 3.5 + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \cdots$$

$$= \frac{35}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6} + \cdots\right)$$

$$= \frac{7}{2} + \frac{23}{10^3}\left(1 + \left(\frac{1}{10^2}\right) + \left(\frac{1}{10^2}\right)^2 + \left(\frac{1}{10^2}\right)^3 + \cdots\right)$$

$$= \frac{7}{2} + \frac{23}{10^3}\sum_{n=1}^{\infty}\left(\frac{1}{10^2}\right)^n$$

$$= \frac{7}{2} + \frac{23}{10^3}\left(\frac{1}{1-\frac{1}{10^2}}\right)$$

$$= \frac{7}{2} + \frac{23}{10^3}\,\frac{10^2}{99}$$

$$= \frac{7}{2} + \frac{23}{990}$$

$$= \frac{1744}{495}$$

Ex Is the series $\sum_{n=1}^{\infty} 2^{2n}\cdot 3^{1-n}$ convergent?

$$\sum_{n=1}^{\infty} 2^{2n}\cdot 3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n \cdot 3^{-(n-1)}$$

$$= \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} \frac{4\cdot 4^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}$$

So this is a geometric series where the common ratio >1

$\Rightarrow$ divergent

$\underline{Ex}$ Compute $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{5^{2n+1}}{7^n}$

Goal: Rewrite as geometric series

$$\sum_{n=1}^{\infty} (-1)^n \frac{5^{2n+1}}{7^n} = \sum_{n=1}^{\infty} \left(\frac{-5}{7}\right) \cdot (-1)^{n-1} \cdot \frac{5^{2n}}{7^{n-1}}$$

$$= -\frac{5}{7} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{7^{n-1}} \cdot (25)^n$$

$$= -\frac{125}{7} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(25)^{n-1}}{7^{n-1}}$$

$$= -\frac{125}{7} \sum_{n=1}^{\infty} \left(\frac{-25}{7}\right)^{n-1}$$

$\uparrow$ geometric series w/ $r = -\frac{25}{7}$

Since $|r| = \frac{25}{7} > 1$, the series diverges

$\underline{Ex}$ Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and compute its sum.

Use partial fractions: $\displaystyle\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$S_n = \sum_{j=1}^{n} \frac{1}{j(j+1)} = \sum_{j=1}^{n} \left(\frac{1}{j} - \frac{1}{j+1}\right)$$

$$= \left(1 - \cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{4}}\right) + \cdots + \left(\cancel{\tfrac{1}{n}} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

(Note: A series where the middle terms cancel in pairs is called a <u>telescoping series</u>.)

Now, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = \underline{1}$

So, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is convergent and $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1.$

<u>Ex</u> Show that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent.

$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$ $\Rightarrow$ If $\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$ converges, then so does $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Now, $\frac{1}{n(n+1)} > \frac{1}{(n+1)^2}$

$\Rightarrow 1 = \sum_{n=1}^{\infty}\frac{1}{n(n+1)} > \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(n+1)^2}$ converges

$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Fact:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

(Basel's Problem)