Other Convergence Tests (§8.4)

Def A series is alternating if it can be written as

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad w/ \ a_n > 0$$

Ex, • $1 - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}}$

The terms alternate between positive and negative.

The Alternating Series Test

If • $a_n > 0$ for every n

• $a_{n+1} \leq a_n$ for all n

• $\lim_{n \to \infty} a_n = 0$

then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$

is convergent.

Ex (The alternating harmonic series)

Recall: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is divergent.

However, by the alternating series test,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

is convergent.

Let's explore this series a bit more.

$$S_2 = (1 - 1/2)$$

$$S_4 = (1 - 1/2) + \underbrace{(1/3 - \tfrac{1}{4})}_{>0} > S_2$$

$$S_6 = (1 - 1/2) + (1/3 - 1/4) + \underbrace{(1/5 - 1/6)}_{>0} > S_4$$

In general, $S_{2n} > S_{2(n+1)}$, so S_{2n} is an increasing sequence.

Now, $S_{2n} = 1 - (1/2 - 1/3) - (1/4 - 1/5) - \cdots - \frac{1}{2n}$

Each bracketed term is >0, so $S_{2n} \leq 1$.

$\Rightarrow$ S_{2n} is increasing and bounded above.

The monotenic sequen theorem $\Rightarrow$ $\{S_{2n}\}_{n=1}^{\infty}$ is convergent.

Write $S = \lim_{n\to\infty} S_{2n}$

For the odd terms:

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}\left(S_{2n} + \frac{1}{2n+1}\right) = \lim_{n\to\infty} S_{2n} + \cancelto{0}{\lim_{n\to\infty}\frac{1}{2n+1}} = S$$

Since $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1} = S$, we have $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = S$.

The proof of the alternating test works in the same way.

$\left(\text{Fact: } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln(2)\right)$