Last time:

Alternating Series Test

If
- $a_n > 0$ for every n
- $a_{n+1} \le a_n$ for all n
- $\lim\limits_{n\to\infty} a_n = 0$

then, the alternating series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$

is convergent.

Ex Show that $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^3+1}$ is convergent.

We want to use alternating series test.

Need to check decreasing:

Let $f(x) = \dfrac{x^2}{x^3+1}$

$$f'(x) = \frac{2x(x^3+1) - 3x^2(x^2)}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2}$$

$f'(x) < 0$ when $x^3 > 2$ or $x > \sqrt[3]{2}$

$\Rightarrow \dfrac{n^2}{n^3+1}$ is decreasing when $n > 1$ (So, the sequence is eventually decreasing)

Now $\lim\limits_{n\to\infty} \dfrac{n^2}{n^3+1} = 0$,

So the alternating series test tells us that

$\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^3+1}$ is convergent.

Def A series $\sum a_n$ is <u>absolutely convergent</u> if the series $\sum |a_n|$ is convergent.

Ex, • $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ is absolutely convergent

• $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is <u>not</u> absolutely convergent

Def A series is called <u>conditionally convergent</u> if it is convergent, but not absolutely convergent.

Ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent

Thm If a series is absolutely convergent, then it is conditionally convergent.

Ex, $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

$$\left|\frac{\cos n}{n^2}\right| \le \frac{1}{n^2}$$

By comparison thm, $\sum_{n=1}^{\infty} \left|\frac{\cos n}{n^2}\right|$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent

$\Rightarrow \sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.

The Ratio Test

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum a_n$ is divergent

- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the Ratio Test is inconclusive.

Ex. Does $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ converge?

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\left(\frac{(n+1)^{n+1}}{(n+1)!}\right) \cdot \left(\frac{n!}{n^n}\right)\right] = \lim_{n\to\infty} \left[\frac{(n+1)(n+1)^n}{(n+1)\cdot n!} \cdot \frac{n!}{n}\right]$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1 \underset{\text{Ratio Test}}{\Rightarrow} \text{Divergent}$$

Now, $$\lim_{n\to\infty} \ln\left[\left(1+\tfrac{1}{n}\right)^n\right] = \lim_{n\to\infty} n \ln\left(1+\tfrac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n} \underset{\text{l'Hôspital}}{=} \lim_{n\to\infty} \frac{\frac{1}{1+1/n} \cdot -1/n^2}{-1/n^2}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

$$\Rightarrow \lim_{n\to\infty} (1+1/n)^n = e^{\lim_{n\to\infty} \ln\left[(1+1/n)^n\right]} = e$$

The Root Test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is abs. convergent
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or is infinite, then $\sum a_n$ diverges.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

Ex Does $\sum_{n=1}^{\infty} \left(\frac{5n-2}{7n+3}\right)^n$ converge?

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{5n-2}{7n+3}\right)^n} = \lim_{n\to\infty} \frac{5n-2}{7n+3} = \frac{5}{7}$$

By the Root Test $\sum_{n=1}^{\infty} \left(\frac{5n-2}{7n+3}\right)^n$ converges.