Power Series (§8.4)

Def A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the coefficients of the series.

A power series may converge for some values of x and diverge for others.

Ex. If we take $c_n = 1$, we get the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

which converges for $-1 < x < 1$ and diverges when $|x| \geq 1$.

Moreover, for each x w/ $|x| < 1$ we know

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } -1 < x < 1$$

Ex For what values of x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Let's apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{x}{n+1} = 0$$

$\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all real numbers

Fact: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

Def A a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series in $(x-a)$ or a power series centered at a.

Ex For what values of x does $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ converge?

Apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{n+1} \cdot \frac{n}{(x-5)^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot (x-5) \right|$$

$$= \left(\lim_{n\to\infty} \left(\frac{1}{1+1/n} \right) \right) \cdot |x-5| = |x-5|$$

The ratio test says that $\sum_{n=0}^{\infty} \frac{(x-5)^n}{n}$ converges when $|x-5|<1$ and diverges when $|x-5|>1$.

So, $|x-5|<1$
$\Rightarrow -1<x-5<1$
$\Rightarrow 4<x<6$

$\Rightarrow$ $\sum_{n=0}^{\infty} \frac{(x-5)^n}{n}$ converges for $4<x<6$ and diverges when $x<4$ or $x>6$.

What about $x=4$ and $x=6$?

When $x=4$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ Converges

$x=6$: $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$ converges for $4 \le x < 6$.

Thm For a given series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only 3 possibilities:

(i) The series Converges only when $x=a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$

In case (iii), the number R is called the radius of convergence.

In case (i) we say $R=0$ and in case (ii) $R=\infty$

The interval of convergence is the interval consisting of all values of x for which the series Converges.

Ex Find the radius of convergence and the interval of convergence for

a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$R = \infty$, Interval: $(-\infty, \infty)$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$

$R = 1$, Interval: $[4, 6)$

Ex Find the radius of convergence and the interval of convergence for

$$\sum_{n=0}^{\infty} \frac{(-4)^n x^n}{\sqrt{n+2}}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-4)^{n+1} x^{n+1}}{\sqrt{n+3}} \cdot \frac{\sqrt{n+2}}{(-4)^n x^n}\right| = \left|\sqrt{\frac{n+2}{n+3}} \cdot (-4x)\right|$$

$$= \left|\sqrt{\frac{1+2/n}{1+3/n}} \cdot 4x\right| \to |4x| \quad \text{as } n \to \infty$$

Series converges for $|4x| < 1$ or $|x| < \frac{1}{4}$

So, $R = \frac{1}{4}$ and series converges for $-\frac{1}{4} < x < \frac{1}{4}$.

Need to check $-1/4$ and $1/4$:

$x = -\frac{1}{4}$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n \cdot \left(\frac{-1}{4}\right)^n}{\sqrt{n+2}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+2}}$ ⇝ divergent

$x = \frac{1}{4}$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n \cdot \left(\frac{1}{4}\right)^n}{\sqrt{n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$ ⇝ Convergent by Alt Series test

$\Rightarrow$ Interval of convergence is $\left(-\frac{1}{4}, \frac{1}{4}\right]$.