Representing Functions as Power Series (§8.6)

Recall: $\boxed{\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1}$

Ex Express $\dfrac{1}{1+4x^3}$ as a power series and find the interval of convergence.

$$\frac{1}{1+4x^3} = \frac{1}{1-(-4x^3)} = \sum_{n=0}^{\infty} (-4x^3)^n = \sum_{n=0}^{\infty} (-4)^n x^{3n}$$

Geometric series $\Rightarrow$ converges for $|-4x^3| < 1$

$\Rightarrow |x^3| < 4$

$\Rightarrow |x| < \sqrt[3]{4}$

$I = (-\sqrt[3]{4}, \sqrt[3]{4})$

Ex Express $\dfrac{1}{5+x}$ as a power series and find its interval of convergence.

$$\frac{1}{5+x} = \frac{1}{5(1+\frac{x}{5})} = \frac{1}{5}\left(\frac{1}{1-(-\frac{x}{5})}\right)$$

$$= \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{x}{5}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} x^n$$

Geometric series $\Rightarrow$ converges when $\left|-\frac{x}{5}\right| < 1$

$\Rightarrow |x| < 5 \Rightarrow I = (-5, 5)$

Ex, Express $\frac{x^6}{5+x}$ as a power series.

$$\frac{x^6}{5+x} = x^6\left(\frac{1}{5+x}\right) = x^6 \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} x^{n+6}$$

Differentiating and Integrating Power Series

Thm If the power series $\sum c_n(x-a)^n$ has radius of convergence R, then the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on $(a-R, a+R)$ and

(i) $f'(x) = \sum_{n=1}^{\infty} n \cdot c_n (x-a)^n$

(ii) $\int f(x)\,dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1}$

The radii of convergence in both (i) and (ii) are R.

We can rewrite (i) and (ii) as

(iii) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right]$$

(iv) $$\int\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty}\left[\int c_n (x-a)^n \, dx\right]$$

* Note: Power series are special: (iii) and (iv) don't always work for an arbitrary series.

Ex Express $\ln(1+x)$ as a power series and find its radius of convergence.

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$$

Now, $$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\Rightarrow \ln(1+x) = \int \frac{1}{1+x}\, dx = \int\left[\sum_{n=0}^{\infty} (-1)^n x^n\right] dx = \sum_{n=0}^{\infty}\left[\int (-1)^n x^n \, dx\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} + C$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n} + C$$

To find C set $x=0$: $\underset{0}{\overset{\parallel}{\ln(1)}} = C$

$$\Rightarrow \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Finally, the radius of convergence is the same as for

$$\frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$$

which is $|-x| < 1$

$\Rightarrow R = 1.$

* Notice $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ converges for $x=1$

This means that

$$\ln(2) = \ln(1+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

But, it actually takes math beyond the scope of the course to show this.

Ex Express arctan x as a power series and find its radius of convergence.

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

So, $\arctan x = \int \frac{1}{1+x^2}\, dx = \int \left[\sum_{n=0}^{\infty} (-1)^n x^{2n}\right] dx$

$$= \sum_{n=0}^{\infty} \left[(-1)^n \int x^{2n}\, dx\right]$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

To find C set $x=0$

$$\underset{\underset{0}{\parallel}}{\arctan(0)} = C$$

$$\Rightarrow \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Like in the previous example, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ converges for $x=1$.

$$\Rightarrow \underset{\underset{\arctan(1)}{\uparrow}}{\frac{\pi}{4}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

<u>Ex</u> Express $\left(\frac{1}{1-7x}\right)^2$ as a power series and find its radius of convergence.

$$\frac{d}{dx}\left(\frac{1}{1-7x}\right) = \frac{7}{(1-7x)^2}$$

$$\parallel$$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} (7x)^n\right) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} 7^n \cdot x^n\right] = \sum_{n=0}^{\infty} (7)^n \left[\frac{d}{dx} x^n\right]$$

Converges for $|7x| < 1$

$\Rightarrow |x| < \frac{1}{7}$

$$= \sum_{n=1}^{\infty} (7)^n \cdot n \cdot x^{n-1}$$

$$\Rightarrow \left(\frac{1}{1-7x}\right)^2 = \frac{1}{7}\sum_{n=1}^{\infty} (7)^n \cdot n \cdot x^{n-1} = \sum_{n=1}^{\infty} 7^{n-1} \cdot n \cdot x^{n-1}$$

The radius convergence is $R = \frac{1}{7}$