Taylor and Maclaurin Series (§8.7)

Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series w/ radius of convergence R.

For $|x-a| < R$, define $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$

- $f(a) = c_0 = 0!\, c_0$

Now, $f'(x) = \sum_{n=1}^{\infty} n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$

- $f'(a) = c_1 = 1!\, c_1$

$f''(x) = \sum_{n=2}^{\infty} n \cdot (n-1) \cdot (x-a)^{n-2} = 2c_2 + 3 \cdot 2\, c_3(x-a) + 4 \cdot 3 \cdot c_4(x-a)^2 + \dots$

- $f''(a) = 2c_2 = 2!\, c_2$

$f'''(x) = \sum_{n=3}^{\infty} n(n-1) \cdot (n-2)(x-a)^{n-3} = 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2\, c_4(x-a) + 5 \cdot 4 \cdot 3 \cdot c_5(x-5)^2 + \dots$

- $f'''(a) = 3 \cdot 2 \cdot c_3 = 3!\, c_3$

Continuing we find $f^{(n)}(a) = n!\, c_n$, or

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Thm If $f(x)$ has a power series representation centered at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R,$$

then

$$c_n = \frac{f^{(n)}(a)}{n!}$$

In other words, if has a power series at a, then

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

This is called the Taylor series of f centered at a.

When $a=0$, we get

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

This is called the Maclaurin series of f.

Ex Find the Maclaurin series of $f(x) = e^x$ and its radius of convergence.

$$\left.\begin{array}{l} f(x) = e^x \Rightarrow f(0) = 1 \\ f'(x) = e^x \Rightarrow f'(0) = 1 \\ f''(x) = e^x \Rightarrow f'(0) = 1 \\ \vdots \\ f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 0 \end{array}\right\} \Rightarrow$$

The Maclaurin series for e^x is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To find the radius of convergence, we use the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \left|\frac{x}{n+1}\right| \to 0 \text{ as } n \to \infty \text{ for every } x$$

$\Rightarrow R = \infty$.

Now, the theorem says if e^x has a power series expansion, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

So, we need a way to check that e^x is equal to its Taylor expansion.

Def. The k^{th} degree Taylor polynomial of f at a is the polynomial

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!} \cdot x^n$$

- The remainder $R_k(x) = f(x) - T_k(x)$.

* Note: The Taylor series for f is $\lim_{k \to \infty} T_k(x)$.

Thm If $f(x) = T_k(x) + R_k(x)$ and

$$\lim_{k \to \infty} R_k(x) = 0$$

for $|x-a| < R$, then f is equal to its Taylor series on the interval $|x-a| < R$.

Thm (Taylor's Formula) If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I there is a number z strictly between x and a such that

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} \cdot (x-a)^{n+1}$$

Let's go back to $f(x) = e^x$.

Taylor's formula tells us that for each n there is a z w/ $0 < z < x$

So that $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} = \dfrac{e^z}{(n+1)!} x^{n+1}$

$\Rightarrow R_n(z) < \dfrac{e^x}{(n+1)!} \cdot x^{n+1}$ since $z < x$ and e^x is increasing.

Now, $\lim_{n \to \infty} \dfrac{x^{n+1}}{(n+1)!} = 0$ (We know since $\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$ converges for every x)

$\Rightarrow \lim_{n \to \infty} R_n(x) = 0$.

We can conclude that

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \quad \text{for all } x}$$