Ex Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$f'''(x) = -\cos x \qquad f'''(0) = -1$$

$$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0 \leftarrow \text{Back to where we started}$$

So, the Maclaurin series is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Now, $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}x^{n+1}$ for some z between 0 and x

$f^{(n+1)}(z)$ is either $\pm\sin(z)$ or $\pm\cos(z) \Rightarrow |f^{(n+1)}(z)| \le 1$

$$0 \le |R_n(x)| \le \left|\frac{x^{n+1}}{(n+1)!}\right|$$

As $\left|\dfrac{x^{n+1}}{(n+1)!}\right| \to 0$ as $n \to \infty$, the Squeeze thm tells us that $\lim_{n\to\infty} |R_n(x)| = 0$ and so $\lim_{n\to\infty} R_n(x) = 0$ for all x.

$$\Rightarrow \boxed{\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \text{ for all } x}$$

Ex Find the Maclaurin Series for $\cos x$.

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty}\left[\frac{d}{dx}\left((-1)^n \frac{x^{2n+1}}{(2n+1)!}\right)\right]$$

$$\Rightarrow \boxed{\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x}$$

Ex Find the Maclaurin series for $x\cos x$.

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n)!}$$

Ex Find the Maclaurin series for $f(x) = (1+x)^k$, where k is any real number.

$f(x) = (1+x)^k$ $\qquad\qquad$ $f(0) = 1$

$f'(x) = k(1+x)^{k-1}$ $\qquad\qquad$ $f'(0) = k$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad\qquad$ $f''(0) = k(k-1)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$ $\qquad\qquad$ $f'''(0) = k(k-1)(k-2)$

$f^{(n)}(k) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$ $\qquad\qquad$ $f^{(n)}(0) = k(k-1)(k-2)\cdots(k-n+1)$

The Maclaurin series for $(1+x)^k$ is $\displaystyle\sum_{n=0}^{\infty}\frac{k(k-1)\cdots(k-n+1)}{n!}x^n$

The numbers $\displaystyle\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!}$ are called <u>binomial coefficients</u>

and $\displaystyle\sum_{n=0}^{\infty}\binom{k}{n}x^n$ a <u>binomial series</u>.

Let's check the radius of convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\binom{K}{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{\binom{K}{n} x^n}\right| = \left|\frac{K\cdot(K-1)\cdots(K-n+1)\cdot(K-n)\cdot x^{n+1} \cdot n!}{K(K-1)\cdots(K-n+1)\cdot x^n \cdot (n+1)!}\right|$$

$$= \left|\frac{K-n}{n+1}\cdot x\right| = \left|\frac{\frac{K}{n}-1}{1+1/n}\right| \cdot |x| \to |x| \text{ as } n\to\infty$$

Ratio Test $\Rightarrow$ binomial series converges for $|x|<1$ and diverges for $|x|>1$.

<u>Thm (Binomial Series)</u> If K is any real number and $|x|<1$, then

$$(1+x)^K = \sum_{n=0}^{\infty} \binom{K}{n} x^n$$

<u>Ex</u> Find the Maclaurin series for $f(x) = \sqrt{9-x}$ and its radius of convergence.

$$f(x) = \sqrt{9-x} = (9-x)^{1/2} = 3\left(1-\frac{x}{9}\right)^{1/2} = 3\left(1+\left(-\frac{x}{9}\right)\right)^{1/2}$$

$$= 3\sum_{n=0}^{\infty} \binom{1/2}{n}\left(-\frac{x}{9}\right)^n = 3\sum_{n=0}^{\infty} \binom{1/2}{n}\cdot\left(\frac{-1}{9}\right)^n \cdot x^n$$

Radius of Convergence:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\binom{K}{n+1}}{(n+1)!} \cdot \left(\frac{x}{9}\right)^{n+1} \cdot \frac{n!}{\binom{K}{n}\left(\frac{x}{9}\right)^n} \right|$$

$$= \left| \frac{K(K-1)\cdots(K-n+1)(K-n)}{K(K-1)\cdots(K-n+1)} \cdot \frac{1}{n+1} \right| \cdot \left|\frac{x}{9}\right|$$

$$= \left| \frac{K-n}{n+1} \right| \cdot \left|\frac{x}{9}\right| = \left| \frac{K/n - 1}{1 + 1/n} \right| \left|\frac{x}{9}\right| \to \left|\frac{x}{9}\right| \text{ as } n \to \infty$$

$\Rightarrow$ Converges for $\left|\frac{x}{9}\right| < 1 \Rightarrow |x| < 9$

$\Rightarrow$ radius of convergence is 9.

Table of Maclaurin Series

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad R=1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R=\infty$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad R=\infty$
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad R=\infty$
- $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \qquad R=1$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \qquad R=1$
- $(1+x)^K = \sum_{n=0}^{\infty} \binom{K}{n} x^n \qquad R=1$