Review: Integration by substitution (§4.5)

Chain rule: $\frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x)$

From the integral viewpoint:

$$\int f'(u(x)) \cdot u'(x)\, dx = f(u(x)) + C \quad \text{(Substitution Rule)}$$

Example: • $\int x^2 \cos(x^3)\, dx$

$u = x^3$, $u'(x) = 3x^2$, so $x^2 = \frac{1}{3} u'(x)$

$$\int x^2 \cos(x^3)\, dx = \int \frac{1}{3} \cos(u(x)) \cdot u'(x)\, dx$$
$$= \frac{1}{3} \sin(u(x)) + C = \frac{1}{3} \sin(x^3) + C$$

• $\int \frac{x}{\sqrt{x^2+1}}\, dx$ (A: $\sqrt{x^2+1} + C$)

§6.1 Integration by parts

Product rule: $\frac{d}{dx}\left[f(x) \cdot g(x)\right] = f(x)\, g'(x) + f'(x) \cdot g(x)$

Integral viewpoint: $\int \left(f(x) \cdot g'(x) + f'(x) \cdot g(x)\right) dx = f(x) \cdot g(x)$

$$\Rightarrow \int f(x)\, g'(x)\, dx + \int f'(x) \cdot g(x)\, dx = f(x) \cdot g(x)$$

$$\Rightarrow \boxed{\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx.}$$

Integration by parts

Usually written as

$$\boxed{\int u\, dv = u \cdot v - \int v \cdot du}$$

Examples

- $$\int \underbrace{x}_{u} \cdot \underbrace{e^x \, dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \, \underbrace{dx}_{du} = xe^x - e^x + C$$

$u = x \qquad dv = e^x \, dx$

$du = dx \qquad v = e^x$

- Try $$\int_0^{\pi} x \sin x \, dx = -x\cos x \Big|_0^{\pi} - \int_0^{\pi} -\cos x \, dx$$

$$= -x\cos x \Big|_0^{\pi} + \sin x \Big|_0^{\pi}$$

$$= (-\pi \cos \pi + 0) + \overset{0}{\sin(\pi)}$$

$$= \pi$$

$u = x \qquad dv = \sin x \, dx$

$du = dx \qquad v = -\cos x \, dx$

- Try $\int x^2 e^x \, dx$

- $$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx$$

$$= x \ln x - x + C$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} dx \qquad v = x$

- $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - \left(e^x \cos x - \int -e^x \sin x \, dx \right)$

$u = \sin x \quad dv = e^x dx \qquad u = \cos x \quad dv = e^x dx$

$du = \cos x \, dx \quad v = e^x \qquad du = -\sin x \quad v = e^x$

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

$$\Rightarrow \quad 2\int e^x \sin x \, dx = e^x (\sin x - \cos x)$$

$$\Rightarrow \quad \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

- Try $\int \arctan x \, dx$