Trig Integrals and Substitutions

Integrals w/ an odd power of a trig function

Recall: $\sin^2\theta + \cos^2\theta = 1$ and $\tan^2\theta + 1 = \sec^2\theta$

$\int \cos x = \sin x + C$

$\int \sin x = -\cos x + C$

Ex. • $\int \cos^3 x \, dx = \int \cos^2 x \cdot \cos x \, dx$

$= \int (1-\sin^2 x) \cos x \, dx$

$= \int (\cos x - \sin^2 x \cdot \cos x) \, dx$

$= \int \cos x \, dx - \int \sin^2 x \cos x \, dx$

$u = \sin x \quad du = \cos x \, dx$

$= \sin x - \int u^2 \, du = \sin x - \frac{u^3}{3} + C$

$= \sin x - \frac{\sin^3 x}{3} + C$

• $\int \sin^5 x \cos^2 x \, dx = \int \sin^4 x \cdot \cos^2 x \cdot \sin x \, dx$

$= \int (\sin^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$

$= \int (1-\cos^2 x)^2 \cdot \cos^2 x \cdot \sin x \, dx$

$u = \cos x \quad du = -\sin x \, dx$

$= \int (1-u^2)^2 \cdot u^2 \cdot (-du)$

$= -\int (1-2u^2+u^4) \cdot u^2 \cdot du$

$= -\int (u^2 - 2u^4 + u^6) \, du$

$= -\frac{u^3}{3} + \frac{2}{5}u^5 - \frac{u^7}{7} + C$

$= -\frac{\cos^3 x}{3} + \frac{2}{5}\cos^5 x - \frac{\cos^7 x}{7} + C$

$$\int \tan^3 x \, dx = \int \tan^2 x \cdot \tan x \, dx$$

$$= \int (\sec^2 x - 1) \cdot \tan x \cdot dx$$

$$= \int \sec^2 x \cdot \tan x \, dx - \int \tan x \, dx$$

$u = \tan x \quad du = \sec^2 x \, dx$

$$= \int u \, du - \ln|\sec x| + C$$

$$= \frac{u^2}{2} - \ln|\sec x| + C$$

$$= \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

$\frac{d}{dx} \tan x = \sec^2 x$

$\int \tan x \, dx = \ln|\sec x| + C$

<u>Theorem</u> $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

<u>proof</u> $\int \sec x \cdot dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$u = \sec x + \tan x \quad du = (\sec x \tan x + \sec^2 x) \, dx$

$$= \int \frac{1}{u} \, du$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

Ex $\int \sec^3 x\, dx = \int \sec^2 x \sec x\, dx$

$u = \sec x \quad dv = \sec^2 x$

$du = \sec x \tan x \quad v = \tan x$

$= \sec x \tan x - \int \sec x \tan^2 x\, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx$

$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$\Rightarrow 2\int \sec^3 x\, dx = \sec x \tan x + \int \sec x\, dx$

$\Rightarrow \int \sec^3 x\, dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$

Ex $\int \tan^3 x \sec^{10} x\, dx = \int \tan^2 x \sec^9 x \cdot \sec x \tan x\, dx$

$= \int (\sec^2 x - 1) \sec^9 x \cdot \sec x \tan x\, dx$

$u = \sec x \quad du = \sec x \tan x\, dx$

$= \int (u^2 - 1) u^9\, du$

$= \int (u^{11} - u^9)\, du$

$= \frac{u^{12}}{12} - \frac{u^{10}}{10} + C$

$= \frac{\sec^{12} x}{12} - \frac{\sec^{10} x}{10} + C$

Integrals w/ only even powers of trig functions

Half-angle identities:

$$\sin^2 x = \frac{1}{2}(1-\cos(2x)) \qquad \cos^2 x = \frac{1}{2}(1+\cos(2x))$$

Ex
$$\int \sin^2 x \, dx = \int \frac{1}{2}(1-\cos(2x))\,dx$$
$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\,dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx = \int \left(\frac{1-\cos(2x)}{2}\right)^2 dx$$
$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \cos^2(2x)\right) dx$$
$$= \frac{1}{4}\int \left(1 - 2\cos(2x) + \frac{1}{2}(1+\cos(4x))\right) dx$$
$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos(2x) + \frac{1}{2}\cos(4x)\right) dx$$
$$= \frac{1}{4}\left[\frac{3}{2}x - \sin(2x) + \frac{1}{8}\sin(4x)\right] + C$$

$$\cdot \int \tan^6 x \sec^4 x \, dx = \int \tan^6 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^6 x \left(\tan^2 x + 1 \right) \sec^2 x \, dx$$

$u = \tan x \quad du = \sec^2 x \, dx$

$$= \int u^6 (u^2 + 1) \, du = \int (u^8 + u^6) \, du$$

$$= \frac{u^9}{9} + \frac{u^7}{7} + C$$

$$= \frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + C$$