§4.2 Continued: Trig Substitution

Recall: $\int \frac{1}{1+x^2}\,dx = \arctan x + C$

Let's prove it:

Trig sub: $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{1+x^2}\,dx = \int \frac{1}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

(Remember, $1+\tan^2\theta = \sec^2\theta$)

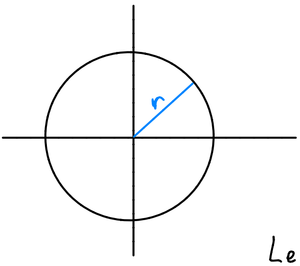
$$= \int \frac{1}{\sec^2\theta}\sec^2\theta\, d\theta = \int d\theta$$

$$= \theta + C$$

Now, $x = \tan\theta$, so $\theta = \arctan x$

$$= \arctan x + C$$

Problem Find the area of the circle w/ radius r.

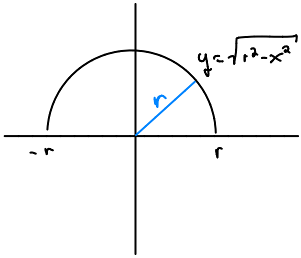


The equation of such a circle is

$$x^2 + y^2 = r^2$$

So, $y = \pm\sqrt{r^2 - x^2}$

Let's focus on $y = \sqrt{r^2 - x^2}$



$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

Substitute $x = r\sin\theta$ $\quad x(0) = r$

$dx = r\cos\theta\, d\theta$ $\quad x(\pi) = -r$

$$= \int_{\pi}^{0} \sqrt{r^2 - r^2\sin^2\theta}\; r\cos\theta\, d\theta$$

$$= r^2 \int_{\pi}^{0} \sqrt{1 - \sin^2\theta}\; \cos\theta\, d\theta$$

$$= r^2 \int_{\pi}^{0} \sqrt{\cos^2\theta}\; \cos\theta\, d\theta$$

$$= r^2 \int_{\pi}^{0} \cos^2\theta\, d\theta$$

$$= r^2 \int_{\pi}^{0} \tfrac{1}{2}(1 + \cos(2\theta))\, d\theta$$

$$= \frac{r^2}{2} \int_{\pi}^{0} (1 + \cos(2\theta))\, d\theta$$

$$= \frac{r^2}{2} \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Big|_{\pi}^{0}$$

$$= \frac{1}{2}\pi r^2$$

Now, this was half a circle, so the area of the circle is $2\left(\frac{1}{2}\pi r^2\right) = \pi r^2$ ✓

In general, the area of an ellipse given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.

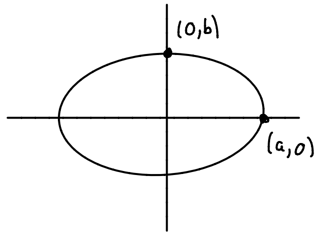


Table of trig subs:

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

Ex. $\displaystyle\int \frac{dt}{t^2\sqrt{4t^2 - 16}}$ $\qquad t = 2\sec\theta$

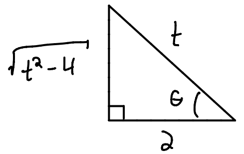
$dt = 2\sec\theta\tan\theta \, d\theta$

$$= \int \frac{2\sec\theta\tan\theta \, d\theta}{4\sec^2\theta\sqrt{16\sec^2\theta - 16}} = \frac{1}{2}\int \frac{\tan\theta}{4\sec\theta\sqrt{\sec^2\theta - 1}} \, d\theta$$

$$= \frac{1}{8}\int \frac{\tan\theta}{\sec\theta\tan\theta} \, d\theta = \frac{1}{8}\int \cos\theta \, d\theta = \frac{1}{8}\sin\theta + C$$

But now, we need to get our answer back in terms of t.

We know $t = 2\sec\theta$ and we need to find $\sin\theta$.



$\sec\theta = \frac{t}{2}$

$\uparrow$

$\frac{\text{Hypotenuse}}{\text{adjacent}}$

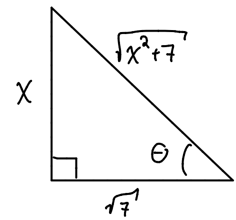
$$\sin\theta = \frac{\sqrt{t^2-4}}{t}$$

Putting it all together,

$$\int \frac{dt}{t^2\sqrt{4t^2-16}} = \frac{1}{8}\frac{\sqrt{t^2-4}}{t} + C$$

Ex $\displaystyle\int \frac{x^3}{\sqrt{x^2+7}}\,dx$

$\tan\theta = \frac{x}{\sqrt{7}}$



$\sec\theta = \frac{\sqrt{x^2+7}}{\sqrt{7}}$

$x = \sqrt{7}\tan\theta \qquad dx = \sqrt{7}\sec^2 x\,dx$

$$= \int \frac{(\sqrt{7}\tan\theta)^3\sqrt{7}\sec^2\theta}{\sqrt{7\tan^2\theta+7}}\,d\theta$$

$$= 7\sqrt{7}\int \frac{\tan^3\theta\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta$$

$$= 7\sqrt{7}\int \tan^3\theta\sec\theta\,d\theta = 7\sqrt{7}\int(\sec^2\theta-1)\tan\theta\sec\theta\,d\theta$$

$$= 7\sqrt{7}\left(\frac{\sec^3\theta}{3} - \sec\theta\right) + C = 7\sqrt{7}\left(\frac{(x^2+7)^{3/2}}{21\sqrt{7}} - \frac{\sqrt{x^2+7}}{\sqrt{7}}\right) + C$$

$$= \frac{(x^2+7)^{3/2}}{3} - 7\sqrt{x^2+7} + C$$