Partial Fractions (§6.3)

Ex, $\dfrac{2}{x-1} - \dfrac{1}{x+2} = \dfrac{2(x+2)-(x-1)}{(x-1)(x+2)} = \dfrac{x+5}{x^2+x-2}$

So, $\underbrace{\displaystyle\int \frac{x+5}{x^2+x-2}\,dx}_{\uparrow} = \displaystyle\int \frac{2}{x-1}\,dx - \int \frac{1}{x+2}\,dx = 2\ln|x-1| - \ln|x+2| + C$

None of our other methods work here.

The goal of partial fractions is to take a ration function $\frac{P(x)}{Q(x)}$ and write as a sum of (hopefully) simpler rational functions.

Ex $\displaystyle\int \frac{2x^3+x}{x-1}\,dx$ If the numerator has a higher degree than the denominator, then we always start with polynomial division.

$$\begin{array}{r l}
 & 2x^2+2x+3 \\
x-1 \,\big) & 2x^3+x \\
 & -(2x^3-2x^2) \\
 & \quad 2x^2+x \\
 & \quad -(2x^2-2x) \\
 & \qquad 3x \\
 & \qquad -(3x-3) \\
 & \qquad\quad 3
\end{array}
\quad \Rightarrow \quad \frac{2x^3+x}{x-1} = 2x^2+2x+3+\frac{3}{x-1}$$

In this case, this is enough:

$$\int \frac{2x^3+x}{x-1}\,dx = \int \left(2x^2+2x+3+\frac{3}{x-1}\right)dx = \frac{2}{3}x^3+x^2+3x+3\ln|x-1|+C$$

Case I: $\int \frac{P(x)}{Q(x)}\,dx$ w/ $Q(x)$ a product of distinct linear factors.

Ex. $\int \frac{x^2+5x-15}{x^3+8x^2+15x}\,dx$

$$x^3+8x^2+15x = x(x^2+8x+15) = x(x+3)(x+5)$$

Goal: $\frac{x^2+5x-15}{x^3+8x^2+15x} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x+5}$

$$x^2+5x-15 = A(x+3)(x+5) + Bx(x+5) + Cx(x+3)$$
$$= Ax^2+8Ax+15A + Bx^2+5Bx + Cx^2+3Cx$$
$$= (A+B+C)x^2 + (8A+5B+3C)x + 15A$$

$\Rightarrow x^2$: $1 = A+B+C$

x: $5 = 8A+5B+3C$

const: $-15 = 15A \Rightarrow A = -1$

$\Rightarrow 1 = -1+B+C \Rightarrow 2 = B+C \Rightarrow C = 2-B$

$5 = -8+5B+3(2-B)$

$13 = 5B+6-3B$

$7 = 2B$

$B = 7/2 \Rightarrow C = 2-\frac{7}{2} = -3/2$

So,

$$\int \frac{x^2+5x-15}{x^3+8x^2+15x}\,dx = \int \frac{-1\,dx}{x} + \int \frac{7/2\,dx}{x+3} + \int \frac{-3/2\,dx}{x+5}$$

$$= -\ln|x| + 7/2\ln|x+3| - 3/2\ln|x+5| + C$$

Case II: $Q(x)$ is a product of linear factors, some of which are repeated

Ex $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

First, polynomial division

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x+1+\frac{4x}{x^3-x^2-x+1}$$

Now, we need to factor x^3-x^2-x+1

It is easy to see $x=1$ is a root.

Some algebra $\rightsquigarrow$ $x^3-x^2-x+1=(x-1)^2(x+1)$

Goal: $$\frac{4x}{x^3-x^2-x+1} = \underbrace{\frac{A}{x-1}+\frac{B}{(x-1)^2}}_{\text{Repeated Roots}}+\frac{C}{x+1}$$

Solve:

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$
$$= (A+C)x^2 + (B-2C)x + (-A+B+C)$$

$$\Rightarrow \left.\begin{array}{l} 0 = A+C \\ 4 = B-2C \\ 0 = -A+B+C \end{array}\right\} \Rightarrow A=1,\ B=2,\ C=-1$$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int \left(x+1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}\right)dx$$
$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$
$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C$$

Case III: $Q(x)$ has an irreducible quadratic factor

Ex $\displaystyle\int \frac{2x^2-x+4}{x(x^2+4)}\,dx$

Goal: $$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$2x^2-x+4 = A(x^2+4) + (Bx+C)x$$
$$= (A+B)x^2 + Cx + 4A$$

$$\left.\begin{array}{l} A+B = 2 \\ C = -1 \\ 4A = 4 \end{array}\right\} \Rightarrow A=1,\ B=1,\ C=-1$$

$$\int \frac{2x^2-x+4}{x(x^2+4)}\,dx = \int\left(\frac{1}{x}+\frac{x-1}{x^2+4}\right)dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

$$\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C}$$

Case IV: $Q(x)$ has repeated irreducible quadratics

Ex $$\frac{2x^2-x+4}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$