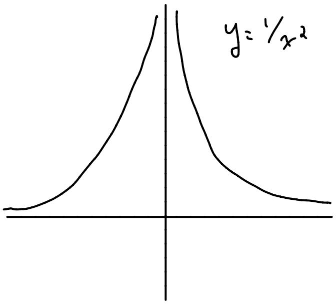
Example Compute $\int_{-1}^{1} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^{1} = \cancel{-2}$

What is the error?



$\frac{1}{x^2}$ is not continuous on the interval $[-1,1]$

So we need to consider what happens at 0:

$$\lim_{a \to 0^+} \int_a^1 \frac{1}{x^2} = \lim_{a\to 0^+} \left(-\frac{1}{x}\Big|_a^1 \right) = \lim_{a \to 0^+} \left(\frac{1}{a} - 1 \right)$$

$$= \infty$$

Improper Integrals of Type 2: Discontinuous Integrands

- If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

- If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

The improper integral $\int_a^b f(x)$ is <u>convergent</u> if the corresponding limit exists and <u>divergent</u> if the limit does not exist.

Example For what values of p is $\int_0^1 \frac{1}{x^p}\,dx$ Convergent?

If $p \neq 1$, the $\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+} \left(\frac{1}{1-p} x^{1-p} \Big|_t^1 \right) = \frac{1}{1-p}\left(1 - t^{1-p}\right)$

$$= \begin{cases} \frac{1}{1-p} & p<1 \\ \infty & p>1 \end{cases}$$

If $p=1$, then $\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \ln x \Big|_t^1 = -\lim_{t\to 0^+} \ln t = \infty$

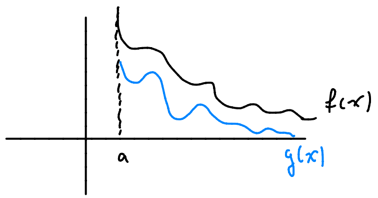
$$\boxed{\int_0^1 \frac{1}{x^p}\,dx \text{ is Convergent if } p<1 \text{ and divergent } \forall\, p\geq 1}$$

Combining w/ last time we see that $\int_0^\infty \frac{1}{x^p}\,dx$ is divergent for every p.

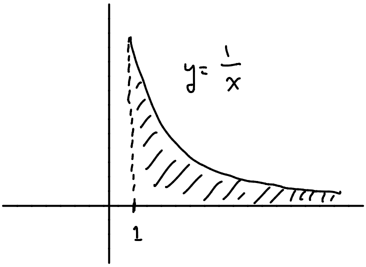
Comparison Theorem

Suppose that f and g are continuous functions w/ $f(x) \geq g(x) \geq 0$ for $x \geq a$.

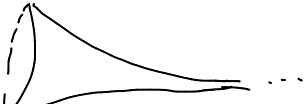
- If $\int_a^\infty f(x)\,dx$ is convergent, then so is $\int_a^\infty g(x)\,dx$
- If $\int_a^\infty g(x)\,dx$ is divergent, then so is $\int_a^\infty f(x)\,dx$



Example (Gabriel's Horn or Torricelli's Trumpet)



Revolve the region under the graph of $y = \frac{1}{x}$ about the x-axis to obtain Gabriel's Horn.



Compute the volume of Gabriel's Horn:

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \pi \left(-\frac{1}{x}\right)\Big|_1^\infty = \pi$$

Compute the surface area of Gabriel's Horn

$$SA = 2\pi \int_1^\infty y\sqrt{1+(y')^2}\, dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx$$

Don't know how to compute this directly, but it is not necessary:

Since $\sqrt{1+\frac{1}{x^4}} \geq 1$ for all x, we have

$$\frac{1}{x}\sqrt{1+\frac{1}{x^4}} \geq \frac{1}{x}$$

$$\Rightarrow \quad SA = \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx \geq \int_1^\infty \frac{1}{x}\, dx = \infty$$

So, Gabriel's Horn has finite volume, but infinite surface area!!!

Example Is $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ convergent?

Break into 3 pieces: $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{-1} e^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{\infty} e^{-x^2}\,dx$

- $\int_{1}^{\infty} e^{-x^2}\,dx$

On $[1,\infty)$, $x^2 > x \Rightarrow e^{x^2} > e^{x} \Rightarrow e^{-x^2} < e^{-x}$

Now, $\int_1^{\infty} e^{-x}\,dx = -e^{-x}\Big|_1^{\infty} = 1$ and thus convergent

$\Rightarrow \int_1^{\infty} e^{-x^2}\,dx$ is convergent

Observe, e^{-x^2} is an even function, so $\int_{-\infty}^{-1} e^{-x^2}\,dx = \int_1^{\infty} e^{x^2}$.

Finally, e^{-x^2} is continuous on $[-1,1]$ so $\int_{-1}^{1} e^{-x^2}\,dx$ is finite,

So $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ is convergent.

"The bell curve"

In fact $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$