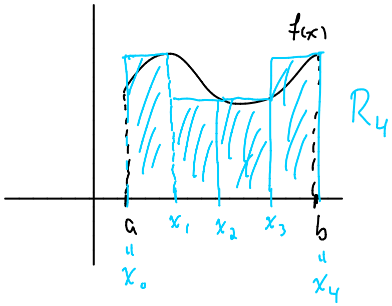
Approximate Integration (§6.5)

The function e^{-x^2} does not have an antiderivative that can be written is closed form, so how do we compute something like $\int_0^1 e^{-x^2}\,dx$? We approximate!

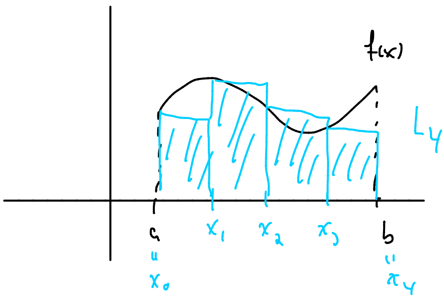
Recall the definition of the integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \underbrace{\boxed{\sum_{i=1}^{n} f(x_i)\,\Delta x}}_{R_n}, \text{ where}$$

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\,\Delta x$$



We can approximate $\int_a^b f(x)\,dx$ by the quantity R_n. The larger we take n to be the better the approximation. (Right Endpoint Approximation)



$$L_n = \sum_{i=0}^{n-1} f(x_i)\,\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L_n$$

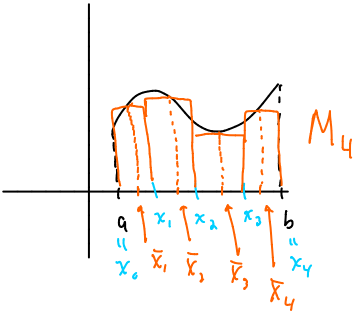
(Left Endpoint Approximation)

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x \,, \text{ where}$$

$$\Delta x = \frac{b-a}{n}$$

$$\bar{x}_i = \frac{1}{2}(x_{i-1}+x_i) = \text{midpoint of } [x_{i-1}, x_i]$$



Trapezoid Rule

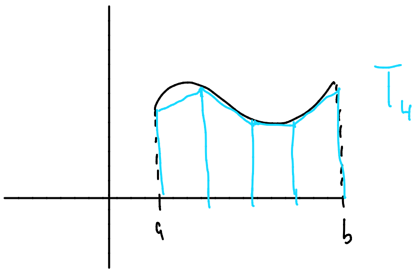
If you average R_n and L_n, then you get the trapezoid rule:

$$\int_a^b f(x)\,dx \approx T_n = \sum_{i=1}^{n} \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]\Delta x$$

$$= \frac{1}{2}\Delta x\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right], \text{ where}$$

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$



Example Approximate $\int_1^2 \frac{1}{x}\,dx$

We'll use $n=4$: $\Delta x = \frac{2-1}{4} = \frac{1}{4}$,

$x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$, $x_3 = \frac{7}{4}$, $x_4 = 2$

$\bar{x}_1 = \frac{9}{8}$, $\bar{x}_2 = \frac{11}{8}$, $\bar{x}_3 = \frac{13}{8}$, $\bar{x}_4 = \frac{15}{8}$

$$R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x = \frac{1}{4}\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right) \approx 0.6345$$

$$L_4 = \sum_{i=0}^{3} f(x_i)\,\Delta x = \frac{1}{4}\left(1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right) \approx 0.7595$$

$$M_4 = \sum_{i=0}^{4} f(\bar{x}_i)\,\Delta x = \frac{1}{4}\left(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right) \approx 0.6912$$

$$T_4 = \sum_{i=1}^{4} \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]\Delta x = \frac{1}{8}\left[1 + 2\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 2\left(\frac{4}{7}\right) + \frac{1}{2}\right] \approx 0.6970$$

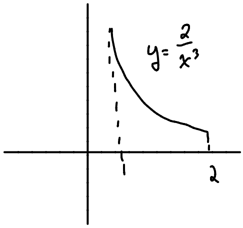
$$\int_1^2 \frac{1}{x}\,dx = \ln x \Big|_1^2 = \ln 2 \approx 0.6931$$

How good are these estimates?

$$\boxed{\begin{array}{l}\underline{\text{Error bounds}} \\ \text{Suppose } |f''(x)| \le K \text{ for } a \le x \le b. \text{ If } E_T \text{ and } E_M \text{ are} \\ \text{the error bounds in the trapezoid and midpoint rules, respectively, then} \\ \qquad |E_T| \le \dfrac{K(b-a)^3}{12n^2} \qquad |E_M| \le \dfrac{K(b-a)^3}{24n^2}\end{array}}$$

Let's apply error bounds to our example:

$f(x) = \frac{1}{x}$



$f'(x) = -\frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

On $[1,2]$ $f''(x) = \frac{2}{x^3}$ is decreasing, so

$$f''(x) \le f''(1) = 2 \quad \text{for } 1 \le x \le 2$$

With $K=2$, $a=1$, $b=2$, $n=4$, the error bounds give

$$|E_T| \le \frac{2(2-1)^3}{12(4)^2} \approx 0.0104$$

$$|E_M| \le \frac{2(2-1)^3}{24(4)^2} \approx 0.0052$$

How big does n need to be do guarantee the trapezoid and midpoint rule are accurate to within 0.001? 0.0001?

$$|E_T| \le \frac{2(2-1)}{12n^2} = \frac{1}{6n^2} < 0.001$$

$$\frac{1000}{6} < n^2 \Rightarrow n > \sqrt{\frac{1000}{6}} \approx 12.91$$

So, $n \ge 13$

$$\frac{1}{6n^2} < 0.0001 \Rightarrow n > \sqrt{\frac{10000}{6}} \approx 40.8$$

$$\Rightarrow n \ge 41$$

$$|E_n| < \frac{2(2-1)}{24n^2} = \frac{1}{12n^2} < 0.001$$

$$\Rightarrow \frac{1000}{12} < n^2$$

$$\Rightarrow n > \sqrt{\frac{1000}{12}} \approx 9.13$$

$$\Rightarrow n \geq 10$$

$$\frac{1}{12n^2} < 0.0001$$

$$\Rightarrow n > \sqrt{\frac{10000}{12}} \approx 28.87$$

$$\Rightarrow n \geq 29$$