Quiz 3 MATH 231
Wednesday, February 22, 2023 Spring 2023

Instructions: This is a take-home quiz. Complete the following problems (on any paper you like) and turn in your solutions in class Monday, February 27. ***You may use your textbook and your homework, but you may not use any other sources.**

Problem 1. Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. Suppose $A\mathbf{x} = \mathbf{b}$ is consistent and that $\mathbf{v}$ and $\mathbf{w}$ are distinct solutions (that is, $\mathbf{v} \neq \mathbf{w}$). Give a nontrivial solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. Justify your answer.

$A(v-w) = Av - Aw = b - b = 0$

$\Rightarrow$ Since $v \neq w$, $v-w$ is a nontrivial solution to $Ax=0$.

Problem 2. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{R}^4$, and let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$. Given that

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

determine if the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent or linearly dependent. Explain your answer.

The matrix equation $Ax=0$ has a free variable, and hence nontrivial Solutions

$\Rightarrow$ $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = 0$ has nontrivial solutions

$\Rightarrow$ v_1, v_2, v_3, v_4 are linearly dependent.

(Turn page over.)

Problem 3. Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in $\mathbb{R}^n$. Show that if $\mathbf{v}_p \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}\}$, then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a subset of $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{p-1}\}$.

Since $v_p \in \operatorname{span}\{v_1, \ldots, v_{p-1}\}$,

$\exists\ c_1, \ldots, c_{p-1} \in \mathbb{R}$ s.t.

$$v_p = c_1 v_1 + c_2 v_2 + \cdots + c_{p-1} v_{p-1}.$$

Now, let $v \in \operatorname{span}\{v_1, \ldots, v_p\}$.

So $\exists\ k_1, \ldots, k_p$ s.t.

$$v = k_1 v_1 + \cdots + k_p v_p.$$

Then $v = k_1 v_1 + \cdots + k_{p-1} v_{p-1} + k_p (c_1 v_1 + \cdots + c_{p-1} v_{p-1})$

$$= (k_1 + k_p c_1) v_1 + (k_2 + k_p c_2) v_2 + \cdots + (k_{p-1} + k_p c_{p-1}) v_{p-1}$$

$\Rightarrow v \in \operatorname{span}\{v_1, \ldots, v_{p-1}\}$

$\Rightarrow \operatorname{span}\{v_1, \ldots, v_p\} \subset \operatorname{span}\{v_1, \ldots, v_{p-1}\}$.