MATH 231: Spring 2023 **Instructor: Nicholas Vlamis**

Wednesday 3/1/2023 Exam 1 110 minutes

Name: Solutions

Instructions.

1. ***Read each problem carefully.*** Make sure you understand what the problem is asking.
2. Unless previously granted permission, you may only use a TI-82, TI-83, TI-84 or scientific calculator.
3. You may use a note sheet, which consists of a single sheet of 8.5" x 11" inch paper. Your note sheet is **not** allowed to contain solutions to problems or proofs of theorems. It will be collected with your exam.
4. No devices other than a writing utensil and calculator may be used.

Question	Points	Score
1	6	
2	8	
3	3	
4	3	
5	3	
6	8	
7	8	
8	4	
9	7	
Total:	50	

Questions

1. $\boxed{\text{6 points}}$ Consider the following system of linear equations:

$$\begin{aligned} x_1 - x_2 - x_3 &= -7 \\ 4x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_2 + 2x_3 &= 8 \end{aligned}$$

(a) Write down the augmented matrix of the linear system.

$$\begin{bmatrix} 1 & -1 & -1 & -7 \\ 4 & 4 & 2 & 0 \\ 0 & 2 & 2 & 8 \end{bmatrix}$$

(b) Write the linear system as a matrix equation.

$$\begin{bmatrix} 1 & -1 & -1 \\ 4 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \\ 8 \end{bmatrix}$$

(c) For the next part, use the fact that the reduced row echelon form of the augmented matrix of the linear system is:

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

How many solutions does the linear system have? If it has solutions, find them all.

1, $x_1 = -3$, $x_2 = 2$, $x_3 = 2$

2. $\boxed{\text{8 points}}$ Let A be a 4×4 matrix and $\mathbf{b} \in \mathbb{R}^4$. Suppose

$$\text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The following questions are about the linear system in the variables x_1, x_2, x_3, and x_4 that is associated to the matrix equation $A\mathbf{x} = \mathbf{0}$, where we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(a) Which variables are the basic (or leading) variables of the linear system? (Note: A is **not** an augmented matrix.)

X_1, X_3

(b) Which variables are the free variables of the linear system?

X_2, X_4

(c) Find the (parameterized) general solution to the linear system.

$X_1 = 2t - 3s$

$X_2 = t$

$X_3 = 7s$

$X_4 = s$

(d) Write the solution to $A\mathbf{x} = 0$ as the span of a collection of vectors in $\mathbb{R}^4$.

$$\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 7 \\ 1 \end{bmatrix} \right\}$$

3. $\boxed{\text{3 points}}$ Let $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -5 & 3 \\ 0 & -1 & 7 \end{bmatrix}$ and let $B = \begin{bmatrix} -2 & 1 & 0 \\ 3 & -7 & 17 \\ 0 & -1 & 7 \end{bmatrix}$.

Find a single elementary row operation that when applied to A yields B. (Make sure you give a clear description of what the operation is.)

Add two times the 3^{rd} row to the 2^{nd} row.

4. $\boxed{\text{3 points}}$ Suppose $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}\right) = T\left(\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right)$$

Find a nontrivial solution to $T(\mathbf{x}) = \mathbf{0}$.

Let $x = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 7 \end{bmatrix}$

Then $T(x) = T\left(\begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}\right) - T\left(\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}\right) = 0$

$\Rightarrow \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix}$ is a nontrivial solution to $T(x) = 0$.

5. $\boxed{\text{3 points}}$ Suppose each of the following matrices is the augmented matrix for a system of linear equations. For each, write down *how many* solutions its corresponding system has. (No computations should be necessary and no work is required.)

(a) $\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$

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6. $\boxed{\text{8 points}}$ Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation satisfying

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\3\\-1\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix}$$

(a) Write $\begin{bmatrix}0\\1\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$.

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}$$

(b) Use the linearity properties of T to find $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$.

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$= \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}1\\3\\-1\end{bmatrix} = \begin{bmatrix}-1\\-2\\3\end{bmatrix}$$

(c) Find the standard matrix for T.

$$\begin{bmatrix}T(e_1) & T(e_2)\end{bmatrix} = \begin{bmatrix}1 & -1\\3 & -2\\-1 & 3\end{bmatrix}$$

(d) Compute $T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)$.

$$T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right) = \begin{bmatrix}1 & -1\\3 & -2\\-1 & 3\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}-3\\-7\\7\end{bmatrix}$$

7. $\boxed{\text{8 points}}$ Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ be vectors in $\mathbb{R}^3$. Let $A = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$, and suppose

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ linearly independent? Explain.

No, the equation $Ax=0$ has ∞-many solutions on account of the second column of corresponding to a free variable in $Ax=0$.

A solution to $Ax=0$ is a sol

(b) Is $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ all of $\mathbb{R}^3$? Explain.

(c) What are the domain and codomain of the matrix transformation T_A?

(d) Is the matrix transformation T_A onto? Explain.

8. $\boxed{\text{4 points}}$ Let $\mathbf{v} \in \mathbb{R}^n$. Show that if $\mathbf{v} \cdot \mathbf{v} = 0$, then $\mathbf{v} = \mathbf{0}$.

9. $\boxed{\text{7 points}}$ Decide whether each of the following statements is TRUE or FALSE (no explanation required).

   (a) A matrix equation of the form $A\mathbf{x} = \mathbf{0}$ always has at least one solution.

   (b) There exists a linear system with exactly three solutions.

   (c) Every linear system with the same number of unknowns and equations has exactly one solution.

   (d) If $T\colon \mathbb{R}^p \to \mathbb{R}^q$ is a linear transformation, then the standard matrix for T has size $q \times p$.

   (e) Any set containing exactly one vector is linearly independent.

   (f) The matrix equation $A\mathbf{x} = \mathbf{b}$ is consistent if every column of A is a pivot column.

   (g) If the rightmost column of an augmented matrix is a pivot column, then the associated linear system is inconsistent.