MATH 231: Spring 2023 **Instructor: Nicholas Vlamis**

Monday 4/3/2023 Exam 2 110 minutes

Name: Solutions

Instructions.

1. ***Read each problem carefully.*** Make sure you understand what the problem is asking.
2. Unless previously granted permission, you may only use a TI-82, TI-83, TI-84 or scientific calculator.
3. You may use a note sheet, which consists of a single sheet of 8.5" x 11" inch paper. Your note sheet is **not** allowed to contain solutions to problems or proofs of theorems. It will be collected with your exam.
4. No devices other than a writing utensil and calculator may be used.

Question	Points	Score
1	5	
2	10	
3	4	
4	5	
5	10	
6	6	
7	3	
8	4	
9	3	
Total:	50	

1. 5 points Let A be a 3×3 matrix with $\det(A) = -3$.

(a) Find $\det(4A)$.

$\det(4A) = 4^3 \det(A) = 64(-3) = -192$

(b) Find $\det(A^{-1})$.

$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{-1}{3}$

(c) Find $\det(A^2)$.

$\det(A^2) = \det(AA) = \det(A)\det(A) = (-3)^2 = 9$

(d) Suppose B is a matrix obtained from A by multiplying the first row of A by 2, the second row by -3, and switching the last two rows of A. Find $\det(B)$.

$\det(B) = (-1)(-3)(2)\det(A) = 6(-3) = -18$

(e) Find $\det(AB)$.

$\det(AB) = \det(A)\det(B) = (-3)(-18) = 54$

2. 10 points Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & -2 \\ 5 & 10 & 2 & -5 \end{bmatrix}$ whose reduced row echelon form is as follows

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What does it mean for a vector v to be in the set null(A)?

$Av = 0$

(b) Is the vector $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} \in \mathbb{R}^4$ in the null space of A?

$A\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 4 & 1 & -2 \\ 5 & 10 & 2 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ Yes

(c) Find a basis for col(A).

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

(d) Find a basis for null(A).

$x_1 = -2t + s$
$x_2 = t$
$x_3 = 0$
$x_4 = s$

$\text{null}(A) = \left\{ \begin{bmatrix} -2t+s \\ t \\ 0 \\ s \end{bmatrix} \middle| s,t \in \mathbb{R} \right\} = \left\{ t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \middle| s,t \in \mathbb{R} \right\}$

$\Rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for null(A)

(e) Find the rank and nullity of A.

rank$(A) = 2$
nullity$(A) = 2$

(f) Determine if the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ is in the column space of A.

$\det\left(\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 5 & 2 & 1 \end{bmatrix} \right) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix}$

$= 1 + (-1)$

$= 0$

$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in \text{col}(A)$ since $\left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is lin dep.

Alternatively:
$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

3. 4 points Let A be an invertible matrix whose inverse is given by $A^{-1} = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 3 & 1 & -5 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.

Let $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

(a) Find a solution to $A\mathbf{x} = \mathbf{b}$.

(b) How many solutions does $A\mathbf{x} = \mathbf{b}$ have? Explain.

4. 5 points Let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ be vectors in $\mathbb{R}^2$.

(a) Show that v_1 and v_2 are a basis for $\mathbb{R}^2$.

(b) Let $\mathcal{B} = \{v_1, v_2\}$. Find the $\mathcal{B}$-coordinates of $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

5. 10 points Answer the following questions. No explanation is necessary.

(a) If A is 3×2 matrix and the matrix product AB is a 3×4 matrix, what is the size of B?

(b) Write down the 3×3 elementary matrix corresponding to the row operation of switching the first and third row.

(c) Suppose A is a 5×7 matrix. What is the value of n for which the column space of A a subspace of $\mathbb{R}^n$?

(d) Suppose A is a 5×7 matrix. What is the value of n for which the null space of A a subspace of $\mathbb{R}^n$?

(e) Suppose A a square matrix that is **not** invertible. What is $\det(A)$?

(f) If A is a 9×7 matrix and the rank of A is 3, then what is the nullity of A?

(g) What is the rank of an invertible 6×6 matrix?

(h) List all the possible values for the nullity of a 3×5 matrix.

(i) Let A be a 8×6 matrix. If A has four pivot columns, what is the size of a basis for the solution set to $Ax = 0$?

(j) Given nonzero vector $u, v \in \mathbb{R}^n$ such that u and v are linearly independent, what is the dimension the subspace $\text{span}\{u, v, u + v\}$?

6. 6 points Let A denote the 3×3 that satisfies $Ae_1 = e_3$, $Ae_2 = e_1$, and $Ae_3 = e_2$.

(a) Write down the matrix A.

(b) Find the determinant of A.

(c) Find the inverse of A.

7. 3 points Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : xy \leq 0 \right\}$, so that W is the union of the second and fourth quadrants in the xy-plane. Determine whether W is a subspace of $\mathbb{R}^2$ or not. Justify your answer.

8. 4 points Use the properties of the determinant to show that if A and B are square matrices such that AB is invertible, then A and B are invertible.

9. 3 points Show that the rows of an invertible $n \times n$ matrix A, viewed as vectors in $\mathbb{R}^n$, are linearly independent. (Hint: think about the transpose of A.)