Homework 1 MATH 301

Solution to graded problem

Exercise 5. Let a and b be nonzero integers.

(1) Prove that the least common multiple of a and b exists.

(2) Prove that if $k \in \mathbb{Z}$ is a common multiple of a and b, then $\operatorname{lcm}(a, b)$ divides k. (Hint: divide k by $\operatorname{lcm}(a, b)$ using the division algorithm.)

Solution. (1) Both ab and $-ab$ are common multiples of a and b, and as at least one of them is positive, a and b have a positive common multiple. Therefore, the set of positive common multiples of a and b is a nonempty subset of the natural numbers. The well-ordering principle implies that this set has a least element, and hence the least common multiple of a and b exist.

(2) Let $k \in \mathbb{Z}$ be a common multiple of a and b, and let $m = \operatorname{lcm}(a, b)$. By the division algorithm, there exists $q, r \in \mathbb{Z}$ such that $k = mq + r$ and $0 \leq r < m$. Observe that as both k and m are common multiples of a and b, we have that $k - mq = r$ is a common multiple of a and b as well. If $r \neq 0$, this would imply that there is a positive common multiple of a and b less than m, which is impossible as m is the least such multiple. Therefore, $r = 0$, and m divides k. $\square$