Homework 3

MATH 301

Due Wednesday, September 27, 2023

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

***Exercise 1.** Let $n \in \mathbb{N}$ with $n > 1$, and let $a \in \mathbb{Z}$.

(a) Prove that if $\gcd(a, n) = 1$ and $b, c \in \mathbb{Z}$ such that $ab = ac \pmod{n}$, then $b = c \pmod{n}$.

(b) Give an example of integers n, a, b, c such that $ab = ac \pmod{n}$ but $b \neq c \pmod{n}$.

***Exercise 2.** Let $n \in \mathbb{N}$.

(a) Prove that $10^n \equiv 1 \pmod{9}$. (There are numerous ways to see this. One way is to use induction.)

(b) (Divisibility by 9) Define $h\colon \mathbb{N} \to \mathbb{Z}$ by

$$h(n) = \sum_{j=0}^{k} a_j,$$

where

$$n = \sum_{j=0}^{k} (a_j \cdot 10^j).$$

In words, $h(n)$ is the sum of the digits of n when written in base 10. For example, if $n = 27301$, then $h(n) = 1 + 0 + 3 + 7 + 2 = 13$. Prove the following statement: Let $n \in \mathbb{N}$, $9 \mid n$ if and only if $9 \mid h(n)$. (Hint: You will have to use Exercise 2(a).)

Exercise 3. Let $n \in \mathbb{N}$.

(a) Prove that $10^n \equiv (-1)^n \pmod{11}$. (Hint: use induction.)

(b) (Divisibility by 11) Define $f\colon \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \sum_{j=0}^{k} (-1)^j a_j,$$

where

$$n = \sum_{j=0}^{k} (a_j \cdot 10^j).$$

In words, $f(n)$ is the alternating sum of the digits of n when written in base 10. For example, if $n = 27301$, then $f(n) = 1 - 0 + 3 - 7 + 2 = -1$. Prove the following statement: Let $n \in \mathbb{N}$, $11 \mid n$ if and only if $11 \mid f(n)$. (Hint: You will have to use Exercise 3(a).)

(Turn page over.)

Exercise 4. Complete the following exercises from Section 3.5 in the course textbook:

#2, 7, 10, 15, *32, 33

(Note that #32 is starred!)

Exercise 5. Let D_4 denote the group of symmetries of a square.

(a) Describe all the elements of D_4. (You do not need to prove you have them all, but do your best. We will do an official count in class, though I'm putting it as a challenge problem below.)

(b) Describe a permutation of the vertices of the square that cannot be obtained via a symmetry of the square. (It might be helpful to view the side lengths of the square as 1 and to use the Pythagorean theorem: $a^2 + b^2 = c^2$, where a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse.)

****Exercise 6.** For $n \in \mathbb{N}$ with $n \geq 3$, let D_n denote the group of symmetries of a regular n-gon. Prove that D_n has $2n$ elements. (We will eventually prove this in class.)