Homework 3

MATH 301

Solution to graded problem

Exercise 4 (#32 in Section 3.5)**.** Show that if G is a finite group of even order, then there is an $a \in G$ such that a is not the identity and $a^2 = e$.

Solution. Let $E = \{g \in G : g^{-1} \neq g\}$. Pairing each element of E with its inverse, we see that E has an even number of elements. As $|G| = |E| + |G \setminus E|$ and both $|G|$ and $|E|$ are even, $G \setminus E$ has an even number of elements. Note if $g \in G \setminus E$, then $g = g^{-1}$, and hence $g^2 = e$. Therefore, to finish, we need to show that $G \setminus E$ contains a non-identity element. We know that $e \in G \setminus E$, so $G \setminus E$ has at least one element. But, $G \setminus E$ is even and hence has at least two elements. Therefore, there exists $a \in G \setminus E$ such that $a \neq e$. As noted above, this means that $a^2 = e$, and a is our desired element of G. □