Homework 5 MATH 301

Solution to graded problem

Exercise 4. Let G be a finite group. Show that there exists an integer N such that $g^N = e$ for every $g \in G$.

Solution. Let $n = |G|$. Let $a_1, a_2, \ldots, a_n$ be a labelling of the elements of G, and let $k_i = |a_i|$. Define $N = k_1 k_2 \cdots k_n$. Then, $g^N = e$ for all $g \in G$ as N is a multiple of $|g|$. □

(I did not grade Exercise 4, but at a quick glance, I did not see many correct proofs, so I included one here.)

Exercise 5. Suppose G is a nontrivial group in which the only two subgroups of G are itself and the trivial subgroup.

(a) Prove that G is cyclic.

(b) Using part (a), prove that G is a finite group of prime order.

Solution. (a) Since G is nontrivial, G contains a non-identity element a. Then, as the only nontrivial subgroup of G is G itself, we must have that $\langle a \rangle = G$; hence, G is cyclic.

(b) If G is infinite, then a has infinite order, and by the same reasoning as above, we must have $\langle a^2 \rangle = G = \langle a \rangle$. But this is impossible: if $(a^2)^k = a$ for some $k \in \mathbb{Z}$, then $a^{2k-1} = e$, which contradicts a having infinite order. Therefore, G has finite order. Now, suppose $|G|$ is not prime, so we can write $|G| = nm$ for some $n, m \in \mathbb{N} \setminus \{1\}$. Then, $|a^n| \leq m < nm$, and hence $\langle a^n \rangle$ is a nontrivial subgroup of G distinct from G. Hence, G must be a finite group of prime order. □