

Homework 6

MATH 301

Solution to graded problem

Exercise 3 (Section 5.4 #25). Prove that in A_n with $n \geq 3$, any permutation is a product of cycles of length 3.

Proof. Let us first show that the product two transpositions is either trivial, a 3-cycle, or a product of two 3-cycles. Let τ_1 and τ_2 be transpositions. Then we can write $\tau_1 = (a b)$ and $\tau_2 = (c d)$ for some $a, b, c, d \in \{1, \dots, n\}$. Up to relabelling, there are three cases: (1) $a = c$ and $b = d$, (2) $a \neq c$ and $b = d$, and (3) both a and b distinct from c and d . In the first case, $\tau_1\tau_2$ is the identity. In the second case, $\tau_1\tau_2 = (a b c)$. In the third case, we have

$$\begin{aligned}\tau_1\tau_2 &= (a b)(c d) \\ &= (a b)(b c)(b c)(c d) \\ &= (a b c)(b c d)\end{aligned}$$

Now, if $\sigma \in A_n$, then σ is a product of a non-zero even number of transpositions (this is true for the identity as well). So, there exists transpositions $\tau_1, \tau_2, \dots, \tau_{2k}$ such that $\sigma = \tau_1\tau_2 \cdots \tau_{2k}$. For $j \in \{1, \dots, k\}$, let $\sigma_j = \tau_{2j-1}\tau_{2j}$. Then, our above argument implies that σ_j is either the identity, a 3-cycle, or a product of two 3-cycles. Therefore, $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ expresses σ as a product of 3-cycles. \square