Homework 11

MATH 301

Due Wednesday, May 10, 2023.

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Prove that every index two subgroup is normal.

Exercise 2. Let $\varphi\colon G \to H$ be a homomorphism of groups. Prove that the kernel of φ is a normal subgroup of G.

Exercise 3. Let $\varphi\colon \mathbb{Z}_7 \to H$ be a homomorphism that is not injective. Determine φ.

Exercise 4. Up to isomorphism, determine the groups H for which there exists a surjective homomorphism from D_4 onto H.

Exercise 5. A 2×2 rotation matrix is a *rotation matrix* of the form

$$R_\theta = \begin{bmatrix} \cos(2\pi\theta) & -\sin(2\pi\theta) \\ \sin(2\pi\theta) & \cos(2\pi\theta) \end{bmatrix}$$

where $\theta \in \mathbb{R}$. The two-dimensional special orthogonal group $SO(2)$ is the set of 2×2 rotation matrices equipped with matrix multiplication.

(a) Prove that $f\colon \mathbb{R} \to SO(2)$ given by $f(\theta) = R_\theta$ is a group homomorphism.

(b) Find the kernel of f.

(c) Prove that $SO(2)$ is isomorphic to $\mathbb{R}/\mathbb{Z}$.

Exercise 6. Complete the following exercises from Section 11.4:

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