Homework 3

MATH 301

Due Wednesday, February 22, 2023

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Exercise 1. Prove that $n^3 \equiv n \pmod 6$ for every integer n.

Exercise 2. Construct the Cayley tables for $(\mathbb{Z}_6, +)$, $(\mathbb{Z}_6, \cdot)$, and $(\mathbb{Z}_7, \cdot)$.

Exercise 3. Let $n \in \mathbb{N}$ with $n > 2$. Prove that there exists $a \in \mathbb{Z}_n$ such that $a^2 = [1] \in \mathbb{Z}_n$ and $a \neq [1]$.

Exercise 4. Let $n \in \mathbb{N}$ with $n > 1$, and let $a \in \mathbb{Z}$.

(a) Prove that if $\gcd(a, n) = 1$ and $b, c \in \mathbb{Z}$ such that $ab = ac \pmod n$, then $b = c \pmod n$.

(b) Give an example of integers n, a, b, c such that $ab = ac \pmod n$ but $b \neq c \pmod n$.

Exercise 5. Let $n \in \mathbb{N}$.

(a) Prove that $10^n \equiv (-1)^n \pmod{11}$. (Hint: use induction.)

(b) Prove that $10^n \equiv 1 \pmod 9$. (There are numerous ways to see this. One way is to use induction as in the previous case.)

Exercise 6 (Divisibility by 9)**.** Define $h\colon \mathbb{N} \to \mathbb{Z}$ by

$$h(n) = \sum_{j=0}^{k} a_j,$$

where

$$n = \sum_{j=0}^{k} (a_j \cdot 10^j).$$

In words, $h(n)$ is the sum of the digits of n when written in base 10. For example, if $n = 27301$, then $h(n) = 1 + 0 + 3 + 7 + 2 = 13$. Prove the following statement: Let $n \in \mathbb{N}$, $9 \mid n$ if and only if $9 \mid h(n)$. (Hint: You will have to use Exercise 5(b).)

Exercise 7 (Divisibility by 11)**.** Define $f\colon \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \sum_{j=0}^{k} (-1)^j a_j,$$

where

$$n = \sum_{j=0}^{k} (a_j \cdot 10^j).$$

In words, $f(n)$ is the alternating sum of the digits of n when written in base 10. For example, if $n = 27301$, then $f(n) = 1 - 0 + 3 - 7 + 2 = -1$. Prove the following statement: Let $n \in \mathbb{N}$, $11 \mid n$ if and only if $11 \mid f(n)$. (Hint: You will have to use Exercise 5(a).)