Homework 5

MATH 301

Due Wednesday, March 15, 2023.

Instructions. *Read the Homework Guide to make sure you understand how to successfully complete the assignment. All claims must be sufficiently justified.*

Definition 1. Let G be a group and let $a \in G$. The *centralizer* of a in G is the set $C_G(a) = \{g \in G : ag = ga\}$.

Exercise 1. Let G be a group and let $a \in G$. Prove that the centralizer of a is a subgroup of G.

Theorem Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{R})$. Then, A is invertible if and only if $\det(A) = ad - bc \neq 0$. Moreover, if A is invertible, then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Definition 2. Let G and G' be groups. A function $h \colon G \to G'$ is a *homomorphism* if $h(ab) = h(a)h(b)$ for all $a, b \in G$, and where the multiplication on the left is in G and the multiplication on the right is in G'.

Exercise 2. Prove that the determinant is a homomorphism from $\mathrm{GL}(2, \mathbb{R})$ to $\mathbb{R}^\times$ (that is, prove that $\det(AB) = \det(A)\det(B)$ for all $A, B \in \mathrm{GL}(2, \mathbb{R})$).

Definition 3. Let $h \colon G \to G'$ be a homomorphism. The *kernel* of h, denoted $\ker(h)$, is the set $\ker(h) = \{g \in G : h(g) = e_{G'}\}$, where $e_{G'}$ is the identity in G'.

Exercise 3. Let $h \colon G \to G'$ be a homomorphism.

(a) Prove that $\ker(h)$ is a subgroup of G.

(b) Use part (a) and Exercise 2 to deduce that $\mathrm{SL}(2, \mathbb{R}) = \{A \in \mathrm{GL}(2, \mathbb{R}) : \det(A) = 1\}$ is a subgroup of $\mathrm{GL}(2, \mathbb{R})$.

Exercise 4. (a) Compute the center of $\mathrm{GL}(2, \mathbb{R})$. (Hint: use the following test matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.)

(b) Compute the center of $\mathrm{SL}(2, \mathbb{R})$.

Exercise 5. Complete the following exercises from Section 3.5 in the course textbook:

36 (this is about D_4), 43, 45, 46, 47, 52 (hint: think about D_3)