Homework 5

MATH 301

Solution to graded problem

Exercise 4. Compute the center of $\mathrm{GL}(2,\mathbb{R})$. (Hint: use the following test matrices $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.)

Solution. Let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ be the 2×2 identity matrix, and let $\mathcal{Z} = \{\lambda I_2 : \lambda \in \mathbb{R}\}$. Then, for any $A = \lambda I_2 \in \mathcal{Z}$ and any $B \in \mathrm{GL}(2,\mathbb{R})$, we have that $AB = (\lambda I_2)B = B(\lambda I_2) = BA$, and hence $A \in Z(\mathrm{GL}(2,\mathbb{R}))$. Hence, $\mathcal{Z} \subset Z(\mathrm{GL}(2,\mathbb{R})$. We will now prove the reverse containment.

Suppose the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in the center of $\mathrm{GL}(2,\mathbb{R})$. Then we have that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}.$$

Since A is in the center, the two products above agree, that is,

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

implying that $c = b$ and $d = a$, so we must have that

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$$

We also have that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a+b & a+b \\ b & a \end{bmatrix}$$

and

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ b & a+b \end{bmatrix}.$$

Again, using that A is in the center, we have that

$$\begin{bmatrix} a+b & a+b \\ b & a \end{bmatrix} = \begin{bmatrix} a & a+b \\ b & a+b \end{bmatrix}$$

implying $a + b = a$, and hence $b = 0$. Therefore,

$$A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI_2 \in \mathcal{Z},$$

showing that $Z(\mathrm{GL}(2,\mathbb{R})) \subset \mathcal{Z}$, and in fact, $\mathcal{Z} = Z(\mathrm{GL}(2,\mathbb{R}))$.

$\square$