Homework 6

MATH 301

Solution to graded problem

Exercise 3 (#30)**.** Suppose that G is a group and let $a, b \in G$. Prove that if $|a| = m$ and $|b| = n$ with $\gcd(m, n) = 1$, then $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Proof. Proof #1: Let $g \in \langle a \rangle \cap \langle b \rangle$, and let $k = |g|$. Viewing g as an element of the cyclic group $\langle a \rangle$ and applying Theorem 4.13, we have that $k \mid m$. Similarly, viewing g as an element of the cyclic group $\langle b \rangle$, it follows that $k \mid n$. But $\gcd(m, n) = 1$, and hence $k = 1$. The only order one element in a group is the identity, so $g = e$. As g was arbitrary, we have that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

Proof #2: Since m and n are relatively prime, there exist integers s and t such that $1 = ms + nt$. Let $g \in \langle a \rangle \cap \langle b \rangle$. Then, $g^m = g^n = e$. So,

$$\begin{aligned} g &= g^1 \\ &= g^{ms+nt} \\ &= (g^m)^s (g^n)^t \\ &= e \end{aligned}$$

Therefore, as g was arbitrary, $\langle a \rangle \cap \langle b \rangle = \{e\}$.

□