MATH 301: Spring 2023 **Instructor: Nicholas Vlamis**

Monday 4/3/2023 Exam 2 110 minutes

Name: Solutions

Instructions.

1. ***Read each problem carefully.*** Make sure you understand what the problem is asking.
2. Proofs can be informal: use of logical symbols and incomplete sentences **are** permitted. However, make sure all statements and logical steps are clear and correct.
3. You are allowed to use notes handwritten by you on the front and back of one 8.5" x 11" sheet of paper. You must turn in your note sheet with the exam.
4. No devices other than a writing utensil may be used.

Question	Points	Score
1	5	
2	4	
3	6	
4	8	
5	5	
6	6	
7	10	
8	6	
Total:	50	

1. [5 points] In each of the following parts, find the order of the element in the given group.

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \mathrm{GL}(3,\mathbb{R})$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Order is 3

(b) $8 \in \mathbb{Z}_{180}$.

$\gcd(8,180) = 4 \Rightarrow |8| = \frac{180}{4} = 45$

$180 = 2^2 \cdot 3^2 \cdot 5$

(c) $(2\ 5\ 6)(1\ 3)(4\ 8\ 9\ 7\ 11)(12\ 13\ 14\ 15) \in S_{15}$.

Order is $\operatorname{lcm}(3,2,5,4) = 60$

2. [4 points] Let G be a group, and let $a, b \in G$ such that $|a| = 9$ and $|b| = 15$. Let g be a non-identity element in the subgroup $\langle a \rangle \cap \langle b \rangle$. What is the order of g? Justify your answer.

$g \in \langle a \rangle \Rightarrow |g| \mid |a| \Rightarrow |g| \mid 9$

$g \in \langle b \rangle \Rightarrow |g| \mid |b| \Rightarrow |g| \mid 15$

$\Rightarrow |g| = 1$ or $|g| = 3$

As g is not the identity, $|g| = 3$.

3. [6 points] (a) Find all the subgroups of $\mathbb{Z}_8$. Justify that your list is complete.

Every subgroup of $\mathbb{Z}_8$ is cyclic.

$\Rightarrow$ $\langle 1\rangle = \langle 3\rangle = \langle 5\rangle = \langle 7\rangle = \mathbb{Z}_8$

$\langle 2\rangle = \langle 6\rangle = \{0,2,4,6\}$ are all the subgroups.

$\langle 4\rangle = \{0,4\}$ $\langle 0\rangle = \{0\}$

(b) Let p be a prime number. Find all the subgroups of $\mathbb{Z}_p$. Justify that your list is complete.

Every subgroup of $\mathbb{Z}_p$ is cyclic.

Let $a \in \mathbb{Z}_p$. Then $|a| \mid p \Rightarrow |a| = 1$ or $|a| = p$.

If $|a| = 1$, then $a = 0$. If $|a| = p$, then $\langle a\rangle = \mathbb{Z}_p$.

$\Rightarrow$ $\mathbb{Z}_p$ has two subgroups, $\{0\}$ and $\mathbb{Z}_p$.

4. [8 points] Recall that, for $n \in \mathbb{N}$, the *group of units modulo* n is the subset $U(n)$ of $\mathbb{Z}_n$ consisting of elements relatively prime to n equipped with multiplication modulo n.

(a) Show that $U(10)$ is cyclic. $U(10) = \{1,3,7,9\}$

$3^0 = 1$
$3^1 = 3$
$3^2 = 9$
$3^3 = 7$

$\Rightarrow \langle 3\rangle = U(10) \Rightarrow U(10)$ is cyclic

(b) Show that $U(12)$ is not cyclic. $U(12) = \{1,5,7,11\}$

$\langle 1\rangle = \{1$

(c) Show that $\mathbb{Q}$ is not cyclic (the group operation is addition).

5. 5 points Let $\sigma \in S_7$ be the permutations defined as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 5 & 1 & 6 & 7 & 4 & 2 \end{pmatrix}$$

(a) Express σ in cycle notation.

(b) Is σ an even or odd permutation? Justify your answer.

(c) Let $\tau = (1254)(123) \in S_7$. Express the permutations $\sigma\tau$ and $\tau\sigma$ in cycle notation.

6. 6 points Prove or disprove: the set $H = \{\sigma \in S_4 : \sigma(2) = 2\}$ is a subgroup of S_4.

7. [10 points] Let $h\colon G \to G'$ be a homomorphism.

(a) Prove that $h(e_G) = e_{G'}$.

(b) Let $g \in G$. Prove that the inverse of $h(g)$ in G' is $h(g^{-1})$.

(c) Prove that $h(G) = \{h(g) : g \in G\}$ is a subgroup of G'.

8. [6 points] Fix an equilateral lateral triangle T. Label the vertices of T by the numbers 1, 2, and 3. Let r be the degree $\frac{2\pi}{3}$ rotation of T and let s be the reflection in the angle bisector at vertex 1. Recall that the elements of the dihedral group D_3 can be expressed as follows: $D_3 = \{1, r, r^2, s, sr, sr^2\}$. Every element of D_3 induces a permutation of the vertices of T. Write down the corresponding permutation in S_3 for each element of D_3.